Newtonian limit

In the *Newtonian limit* (the limit of weak fields, $\phi \ll c^2$, and slow motions, $v \ll c$) General Relativity reduces to Newtonian gravitation. That is: i) the Einstein's field equation reduces to Poisson equation for Newtonian gravitational potential ϕ; and ii) the geodesic equation reduces to Newton's equation of motion.

Newtonian gravitation

Newton's law of universal gravitation

The Newton's law of universal gravitation (published in 1687) states that two particles with masses m and M, located at a relative distance r, attract each other with the force

$$F = G\frac{mM}{r^2}, \tag{1}$$

where $G \approx 6.674 \times 10^{-11}\mathrm{N} \cdot \mathrm{m}^2\mathrm{kg}^{-2}$ is the gravitational constant. The latter was first measured by Henry Cavendish in 1798.

Potential field formulation of Newtonian gravitation

The force (1) is *conservative* and allows potential formulation: the body M creates a *gravitational potential* ϕ,

$$\phi(\vec{r}) = -\frac{GM}{r}\,, \tag{2}$$

in which the particle m acquires a potential energy $m\phi$ with the corresponding force

$$\vec{F} = -m\nabla\phi\,. \tag{3}$$

The potential (2) satisfies the Poisson equation[1]

$$\nabla^2\phi(\vec{r}) = 4\pi GM\delta(\vec{r})\,. \tag{4}$$

If instead of a single point-mass with mass density $M\delta(\vec{r})$ there is a distribution of masses with density $\mu(\vec{r})$, the gravitational potential created by this distribution of masses satisfies the Poisson equation

$$\nabla^2\phi = 4\pi G\mu\,. \tag{5}$$

It is this equation that the Einstein's equation must reduce to in the slow-motion weak-field limit.

Metric in Newtonian limit

Equation (3) can be cast into a variational form with the action

$$\begin{aligned} S &= \int dt\left(\frac{1}{2}mv^2 - m\phi - mc^2\right) \\ &= -mc\int dt\left(c - \frac{v^2}{2c} + \frac{\phi}{c}\right). \end{aligned} \tag{6}$$

Comparing with $S = -mc\int ds$ we get (squaring and dropping terms negligible in the limit $c \to \infty$)

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right)c^2dt^2 - d\vec{r}^2\,. \tag{7}$$

Thus in the Newtonian limit the metric tensor can be approximated by $g_{ab} = \eta_{ab} + h_{ab}$, where η_{ab} is the Minkowski metric tensor and h_{ab} is a small correction with the g_{00} component given as

$$g_{00} = 1 + \frac{2\phi}{c^2}\,, \tag{8}$$

where ϕ satisfies the Poisson equation (5).

[1] $\nabla^2\frac{1}{r} = -4\pi\delta(\vec{r})$.

Newtonian limit of general relativity

For a distribution of (otherwise non-interacting) masses with mass density $\mu(\vec{r})$ the stress-energy-momentum tensor is

$$T_{ab} = \mu c^2 u_a u_b \,. \tag{9}$$

In the Newtonian limit, where all fields are weak and all velocities are small, $u_a = \{1, 0, 0, 0\}$, only the $_{00}$ component of the stress-energy-momentum tensor is non-vanishing,

$$T_{00} = \mu c^2 \tag{10}$$

Therefore we shall only consider the $_{00}$ component of the Einstein's equation,

$$R_{00} = \kappa(T_{00} - \frac{1}{2} g_{00} T) = \frac{1}{2}\kappa\mu c^2 \,. \tag{11}$$

In the slow-weak limit all second order terms and temporal derivatives must be neglected altogether. The $_{00}$ component of the Ricci tensor then reduces to

$$R_{00} \mathrel{\mathring{=}} R^a_{0a0} = \Gamma^\alpha{}_{00,\alpha} \tag{12}$$

where the Greek symbols run over $1, 2, 3$.

Assuming $g_{00} = 1 + 2\phi/c^2$, where ϕ/c^2 is a small correction, and dropping the temporal derivatives, the Christoffel symbols become

$$\Gamma_{\alpha 00} = -\frac{1}{c^2}\phi_{,\alpha} \,, \tag{13}$$

The Ricci tensor in the same limit is given as

$$R_{00} = -\frac{1}{c^2}\eta^{\alpha\beta}\phi_{,\alpha\beta} \equiv \frac{1}{c^2}\nabla^2\phi \,, \tag{14}$$

The Einstein equation thus turns into the Poisson's equation

$$\nabla^2\phi = \frac{1}{2}\kappa c^4 \mu \,, \tag{15}$$

which is equivalent to the Newtonian theory if we put

$$\kappa = \frac{8\pi G}{c^4} \,. \tag{16}$$

Gravitational waves

The Einstein equation predicts *gravitational waves* — the vacuum solutions in the form $f(ct - x)$ — which propagate with the speed of light and carry energy and momentum.

Weak gravitational waves in vacuum

In a weak gravitational field the space-time is almost flat and the metric tensor g_{ab} equals the flat metric η_{ab} plus a small correction $h_{ab} \ll 1$,

$$g_{ab} \;=\; \eta_{ab} + h_{ab} \,. \tag{17}$$

The Riemann tensor to the lowest order in h_{ab} is

$$R_{abcd} = \frac{1}{2}\left(h_{ad,bc} + h_{bc,ad} - h_{ac,bd} - h_{bd,ac}\right) . \tag{18}$$

The Ricci tensor to the lowest order,

$$R_{bd} = \eta^{ac} R_{abcd} = \frac{1}{2}\left(\bar{h}^{a}_{b,ad} + \bar{h}^{a}_{d,ab} - h_{bd,a}^{;a}\right) , \tag{19}$$

where

$$\bar{h}_{ab} \doteq h_{ab} - \frac{1}{2} h \eta_{ab} , \tag{20}$$

and $h \doteq h^{a}_{a}$.

If we choose our coordinates such that (see the exercise)

$$\bar{h}^{a}_{b,a} = 0 , \tag{21}$$

(which is simetimes called *the Lorentz gauge*) the Ricci tensor becomes

$$R_{ab} = -\frac{1}{2} h_{ab,c}^{;c} , \tag{22}$$

and the vacuum Einstein's equation, $R_{ab} = 0$, turns into the ordinary wave equation,

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) h_{ab} = 0 .$$

Detection of gravitational waves

The intensity of gravitational radiation by a system of slowly moving bodies is determined by its quadrupole moment $D_{\alpha\beta}$,

$$-\frac{dE}{dt} = \frac{G}{45c^5} (\dddot{D}_{\alpha\beta})^2 . \tag{23}$$

This is a rather weak intensity, and even the waves from the most potent sources—mergers of black holes or neutron stars—have a very small amplitude (10^{-21}) when they reach Earth. Nevertheless the interferometer-type detector LIGO managed in 2015 to detect a gravitational wave from a merger of two black holes about a billion light-years away.

The interferometer-type detectors work like this: consider a gravitational wave moving in the x-direction and having the h_{yz}-form (see the exercise),

$$h_{ab \neq yz,zy} = 0 \ , \ \ h_{yz} = f(t - x) \ , \tag{24}$$

and consider an interferometer in the yz-plane with its legs stretching from the origin to the points $(-dy, dz)$ and (dy, dz). The lengths of its legs, dl_1^2 and dl_2^2, will be modified[2] by a passing gravitational wave as

$$dl_1^2 = dy^2 + dz^2 + 2h_{yz} dy dz \ , \tag{25}$$

$$dl_2^2 = dy^2 + dz^2 - 2h_{yz} dy dz \ . \tag{26}$$

If the legs are long enough and the interferometer is sensitive enough (and the merging black holes are big enough) these minuscule changes can be (and have been) detected.

Exercises

1. Argue that $\nabla^2 \frac{1}{r} = -4\pi\delta(\vec{r})$.

   Hints:

   (a) argue that $\nabla^2 \frac{1}{r} = 0$ everywhere except for the origin;

[2] We tacitly assume that the effect of the gravitational wave on the wavelengt of the laser beam in the interferometer is of the second order.

(b) using the divergence theorem,

$$\int_V (\vec{\nabla}\vec{f})dV = \oint_{\partial V} (\vec{f}\cdot\vec{n})dS\,,$$

establish the factor 4π.

2. Consider the metric in newtonian limit,

$$ds^2 = \left(1+\frac{2\phi}{c^2}\right)c^2dt^2 - d\vec{r}^2\,.$$

Show that the geodesic equation, $Du^a = 0$, is equivalent to the Newton's equation of motion, $d\vec{v}/dt = -\nabla\phi$.

3. Assume that the graviational field is weak and that the metric tensor g_{ab} equals the Minkowski metric tensor η_{ab} plus a small correction h_{ab} where $|h_{ab}| \ll 1$,

$$g_{ab} = \eta_{ab} + h_{ab}\,.$$

(a) Calculate the Riemann tensor and the Ricci tensor in the lowest order in h_{ab}.

(b) Write down the Einstein equation in vacuum in the same order (the linearized Einstein equation in vacuum).

4. Show that in the weak field limit, $g_{ab} = \eta_{ab}+h_{ab}$, it is always possible to find an infinitesimal coordinate transformation

$$x^a \to x'^a = x^a + \epsilon^a$$

such that

$$(h'^a_b - \frac{1}{2}h'\delta^a_b)_{,a} = 0\,.$$

You might need to show first that under this infinitesimal transformation

$$\delta g_{ab} = -\epsilon_{a;b} - \epsilon_{b;a}\,.$$

5. In the (second) Nordström's theory of gravitation the metric tensor is given as $g_{ab} = e^{2\phi/c^2}\eta_{ab}$ where the function ϕ is determined by the field equation $R = \kappa T$ (where $R = g^{ab}R_{ab}$ and $T = g_{ab}T^{ab}$).

(a) Argue that in the Newtonian limit this theory reproduces Newtonian gravitation (with the appropriate choice of the constant κ).

(b) Argue that this theory predicts gravitational waves.

6. In the weak field limit show that the metric tensor in the form $g_{ab} = \eta_{ab} + h_{ab}$, where $h_{ab\neq yz,zy} = 0$, $h_{yz} = f(t-x)$, and f is an arbitrary function, satisfies the linearized Einstein equation in vacuum.

7. In the weak field limit show that the metric tensor in the form $g_{ab} = \eta_{ab} + h_{ab}$, where $h_{yz} = A\sin\omega(t-x)$, $h_{tt} = 2f(t-x)$, $h_{tx} = -f(t-x)$, all other $h_{ab} = 0$, f is an arbitrary function, satisfies the linearized Einstein equation in vacuum.

8. (Not for exam?) Consider the metric in newtonian limit in the gravitational potential at the surface of the Earth in the Earth's frame. Make a coordinate transformation to the frame where the Christoffel symbols are zero. Argue that it is a free falling frame. Hint: the general transformation to a locally-inertial frame at the origin is given as

$$x'^a = x^a + \frac{1}{2}\Gamma^a{}_{bc}(0)x^bx^c\,.$$

9. (Not for exam?) Consider dust (non-interacting incoherent matter) – a good approximation for our universe just at the moment. Argue, that its stress-energy-momentum tensor is given as

$$T^{ab} = \mu u^a u^b \,,$$

where μ is the mass density of the dust measured by a co-moving observer, and $u^a = dx^a/ds$. Use the following strategy:

   (a) Argue, that it is a generally covariant tensor.

   (b) Argue that in special relativity T^{00} is the energy density.

   (c) Argue that in special relativity this tensor satisfies the equations,

   $$T^{ab}{}_{,b} = 0 \,,$$

   by arguing that the 0-component, $T^{0b}{}_{,b} = 0$, represents the energy conservation law,

   $$\frac{\partial \epsilon}{\partial t} + \vec{\nabla} \cdot (\epsilon \vec{v}) = 0 \,,$$

   and that the spatial components, $T^{\alpha b}{}_{,b} = 0$, where $\alpha = 1, 2, 3$ represent the Navier-Stokes equation for the free dust motion (no stress),

   $$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = 0 \,.$$