Einstein equation

In this section we shall introduce the *Einstein field equation*: the equation that relates the geometry of space-time with the distribution of matter within it.

In electrodynamics the field equation is given by the inhomogeneous Maxwell equation,

$$F^{ab}_{;a} = 4\pi j^b \,, \tag{1}$$

which relates the (covariant derivatives of the) electromagnetic field F^{ab} with its source, the electric current j^b. One might expect the corresponding field equation of general relativity to have a similar form: the source of the gravitational field at the right-hand-side and some (covariant) combination of the derivatives of the gravitational field at the left-hand-side.

The source of the gravitational field in general relativity is the stress-energy-momentum of matter, T_{ab}. The corresponding field equation should then have the form

$$G_{ab} = \kappa T_{ab} \tag{2}$$

where κ is a constant and where G_{ab} is a certain tensor made of the derivatives of the gravitational field (the metric tensor). Since T_{ab} is a symmetric divergenceless tensor, so should G_{ab} be as well. In the following section we shal derive the simple such tensor.

Riemann curvature tensor

In this section we shall introduce a tensor associated with the curvature of space—the *Riemann curvature tensor*—and try to build a suitable G_{ab} out of it.

The Riemann tensor is usually introduced in one of the two ways: i) as a commutator of covariant derivatives; or ii) via a parallel transport of a vector around a small loop.

Commutator of covariant derivatives

One way to introduce the Riemann tensor is to consider the commutator of covariant derivatives,

$$A_{b;cd} - A_{b;dc} \,. \tag{3}$$

It is apparently a covariant object which is identically zero in a flat space[1] and is (generally) not identically zero in a curved space.

Let us calculate it. First the double covariant derivative,

$$\begin{aligned} A_{b;cd} &= (A_{b;c})_{;d} = (A_{b;c})_{,d} - \Gamma^e{}_{bd} A_{e;c} - \Gamma^e{}_{cd} A_{b;e} \\ &= (A_{b,c} - \Gamma^a{}_{bc} A_a)_{,d} - \Gamma^e{}_{bd} \left(A_{e,c} - \Gamma^a{}_{ec} A_a\right) - \Gamma^e{}_{cd} \left(A_{b,e} - \Gamma^a{}_{be} A_a\right) \\ &= A_{b,cd} - \Gamma^a{}_{bc,d} A_a - \Gamma^a{}_{bc} A_{a,d} - \Gamma^e{}_{bd} A_{e,c} + \Gamma^e{}_{bd} \Gamma^a{}_{ec} A_a - \Gamma^e{}_{cd} A_{b,e} + \Gamma^e{}_{cd} \Gamma^a{}_{be} A_a \,. \end{aligned} \tag{4}$$

Now the commutator,

$$A_{b;cd} - A_{b;dc} = \left(-\Gamma^a{}_{bc,d} + \Gamma^a{}_{bd,c} - \Gamma^e{}_{bc} \Gamma^a{}_{ed} + \Gamma^e{}_{bd} \Gamma^a{}_{ec}\right) A_a \,. \tag{5}$$

The tensor in the parentheses is called the Riemann curvature tensor $R^a{}_{bcd}$,

$$R^a{}_{bcd} \doteq -\Gamma^a{}_{bc,d} + \Gamma^a{}_{bd,c} - \Gamma^e{}_{bc} \Gamma^a{}_{ed} + \Gamma^e{}_{bd} \Gamma^a{}_{ec} \tag{6}$$

The Riemann tensor thus determines the commutator of covariant derivatives

$$A_{b;cd} - A_{b;dc} = R^a{}_{bcd} A_a \,. \tag{7}$$

[1] Indeed in a flat space we can make a global transformation to Cartesian coordinates where this commutator is obviously zero.

Parallel transport around infinitesimal loop

An alternative way to introduce the Riemann tensor is to consider a parallel transport of a vector (a transport with $DA^a = 0$). Under an infinitesimal parallel transport of a vector A^a along a path dx, the components of the vector (generally) change,

$$dA^a\big|_{DA^a=0} = -\Gamma^a{}_{bc} A^b dx^c \,. \tag{8}$$

The change of a vector along an infinitesimal closed contour, $\Delta A^a = \oint dA^a$, is a covariant quantity since it is the difference between two vectors at the same point. It must be proportional to the vector itself and the area of the surface enclosed by the contour.

Let us calculate this change using a simple rectangular contour with sides dx and dx'. The vector A^a after a parallel transform along dx and then dx' turns into

$$A^a(x + dx + dx') = A^a - \Gamma^a{}_{bc} A^b dx^c - \left(\Gamma^a{}_{bc} + \Gamma^a{}_{bc,d} dx^d\right)\left(A^b - \Gamma^b{}_{df} A^d dx^f\right) dx'^c \,. \tag{9}$$

Similarly, after a parallel transform first along dx' and then along dx

$$A^a(x + dx' + dx) = A^a - \Gamma^a{}_{bc} A^b dx'^c - \left(\Gamma^a{}_{bc} + \Gamma^a{}_{bc,d} dx'^d\right)\left(A^b - \Gamma^b{}_{df} A^d dx'^f\right) dx^c \,. \tag{10}$$

Subtracting these two and leaving only the lowest order terms gives (after some index renaming)

$$\begin{aligned} \Delta A^a &= A^a(x + dx' + dx) - A^a(x + dx + dx') \\ &= \left(\Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}\right) A^b dx^c dx'^d \\ &\mathrel{\hat{=}} R^a{}_{bcd} A^b dx^c dx'^d \,. \end{aligned} \tag{11}$$

The tensor in parentheses is called the Riemann curvature tensor $R^a{}_{bcd}$,

$$R^a{}_{bcd} \mathrel{\hat{=}} \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}. \tag{12}$$

It follows from (11) that if the Riemann tensor vanishes, the vector A^a does not change when parallelly transported along an arbitrary infinitesimal closed path. Also after a parallel transport from x to $x + \delta x$ the vector $A^a(x + \delta x)$ is independent on which path is taken. The space where Riemann tensor vanishes everywhere is flat.

Properties of the curvature tensor

1. Antisymmetry:
$$R_{abcd} = -R_{bacd} = -R_{abdc} \,. \tag{13}$$

2. Interchange symmetry:
$$R_{abcd} = R_{cdab} \,. \tag{14}$$

3. Algebraic Bianchi identity:
$$R_{abcd} + R_{acdb} + R_{adbc} = 0 \,. \tag{15}$$

4. Differential Bianchi identity:
$$R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0, \tag{16}$$

Ricci tensor and Ricci scalar

The Ricci tensor is obtained by a contraction of the Riemann tensor over a pair of indexes,

$$R_{ab} \mathrel{\hat{=}} R^d{}_{adb} \,. \tag{17}$$

It is symmetric,

$$R_{ab} = R_{ba} \,. \tag{18}$$

The Ricci scalar (or scalar curvature) is the simplest curvature scalar in a Riemannian geometry,

$$R \mathrel{\hat{=}} g^{ab} R_{ab} \,. \tag{19}$$

Contraction of the differential Bianchi identity over bc and ad gives

$$R^b_{a;b} = \frac{1}{2} R_{,a} \,, \tag{20}$$

from which it follows, that the (Einstein) tensor

$$G_{ab} \mathrel{\hat{=}} R_{ab} - \frac{1}{2} R g_{ab} \tag{21}$$

is symmetric and divergenceless

$$G^{ab}_{;b} = 0 \,. \tag{22}$$

The gravitational field equations might therefore have the form

$$G^{ab} = \kappa T^{ab} \,, \tag{23}$$

where the constant κ is related to the used units of mass (energy) like the 4π in the Maxwell equations in Gauss units.

In the following section we shall derive this equations from an action.

Hilbert-Einstein action and Einstein equation

The simplest action for gravitation is the Hilbert-Einstein action – the integral over the Ricci scalar,

$$S_g = -\frac{1}{2\kappa} \int R \sqrt{-g} d\Omega, \tag{24}$$

where κ is a constant (to be determined from the experiment).

Variation of $R\sqrt{-g}$ with respect to δg^{ab} gives

$$\delta \left(R\sqrt{-g} \right) = \delta \left(R_{ab} g^{ab} \sqrt{-g} \right) = R_{ab} \delta g^{ab} \sqrt{-g} + R \delta \sqrt{-g} + g^{ab} \sqrt{-g} \delta R_{ab} \,. \tag{25}$$

The last term can be shown to be a divergence[2] which does not contribute to variation. In the second term we have[3]

$$\delta \sqrt{-g} = -\frac{1}{2} \frac{1}{\sqrt{-g}} g g^{ab} \delta g_{ab} = -\frac{1}{2} \sqrt{-g} g_{ab} \delta g^{ab}. \tag{31}$$

[2]The variation $\delta R^a{}_{bcd}$ of the Riemann tensor is given as

$$\delta R^a{}_{bcd} = \delta \Gamma^a{}_{bd,c} - \delta \Gamma^a{}_{bc,d} + \delta \Gamma^a{}_{fc} \Gamma^f{}_{bd} + \Gamma^a{}_{fc} \delta \Gamma^f{}_{bd} - \delta \Gamma^a{}_{fd} \Gamma^f{}_{bc} - \Gamma^a{}_{fd} \delta \Gamma^f{}_{bc} \,. \tag{26}$$

The difference $\delta \Gamma^a{}_{bc}$ is a tensor (indeed, $\delta DA^a = \delta \Gamma^a_{bc} A^b dx^c$), therefore

$$(\delta \Gamma^a{}_{bc})_{;d} = (\delta \Gamma^a{}_{bc})_{,d} + \Gamma^a{}_{ed} \delta \Gamma^e{}_{bc} - \Gamma^e{}_{ad} \delta \Gamma^a{}_{ec} - \Gamma^e{}_{dc} \delta \Gamma^a{}_{de} \,. \tag{27}$$

The variation of the Riemann tensor can then be written as

$$\delta R^a{}_{bcd} = (\delta \Gamma^a{}_{bd})_{;c} - (\delta \Gamma^a{}_{bc})_{;d} \,. \tag{28}$$

The variation of the Ricci tensor is now given as

$$\delta R_{bd} = \delta R^a{}_{bad} = (\delta \Gamma^a{}_{bd})_{;a} - (\delta \Gamma^a{}_{ba})_{;d} \,. \tag{29}$$

Finally,

$$\sqrt{-g} g^{bd} \delta R_{bd} = \sqrt{-g} \left(g^{bd} \delta \Gamma^a{}_{bd} - g^{ba} \delta \Gamma^d{}_{bd} \right)_{;a} = \left(\sqrt{-g} g^{bd} \delta \Gamma^a{}_{bd} - \sqrt{-g} g^{ba} \delta \Gamma^d{}_{bd} \right)_{,a} \blacksquare \tag{30}$$

[3]using

$$dg = g g^{ab} dg_{ab} = -g g_{ab} dg^{ab} \,.$$

Thus the variation of the Hilbert action is

$$\delta S_g = -\frac{1}{2\kappa}\int \left(R_{ab} - \frac{1}{2}Rg_{ab}\right)\delta g^{ab}\sqrt{-g}d\Omega. \tag{32}$$

As we have shown above, the variation of the matter action with respect to the metric tensor is determined by the stress-energy-momentum tensor,

$$\delta S_m = \frac{1}{2}\int T_{ab}\delta g^{ab}\sqrt{-g}d\Omega\,.$$

Combining the variations of the matter action and the gravitational action into the variational principle

$$\delta(S_m + S_g) = 0 \tag{33}$$

gives the famous Einstein's gravitational field equation,

$$R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab}\,. \tag{34}$$

The constant κ is related to the Newtonian grafitational constant G (comparing the non-relativistic limit of the Einstein's equation with Newtonian theory of gravitation gives $\kappa = 8\pi G/c^4$).

Note that the covariant divergence of the left-hand side vanishes, due to the Bianchi identity. Therefore the covariant divergence of the right-hand side, the stress-energy-momentum tensor, should also vanish. The latter, however, contains the equations of motion for the matter. Therefore the Einstein equation determines simultaneously both the gravitational field created by matter and the motion of the matter in this gravitational field.

Exercises

1. Argue that—due to the symmetry properties of the Riemann tensor—the Ricci tensor is the only 2-index tensor that can be obtained from the Riemann tensor by contracting its indices.

2. Argue that the Einstein's field equation can be also written as

$$R_{ab} = \kappa(T_{ab} - \frac{1}{2}Tg_{ab})\,,$$

   where T is the trace of the stress-energy tensor of matter.

3. Prove[4] that the Riemann tensor with lowered index, $R_{abcd} \mathrel{\hat{=}} g_{ae}R^e_{\ bcd}$, can be written as

$$R_{abcd} = -\Gamma_{abc,d} + \Gamma_{abd,c} - \Gamma_{eac}\Gamma^e_{\ bd} + \Gamma_{ead}\Gamma^e_{\ bc}\,,$$

   and as

$$R_{abcd} = \frac{1}{2}\left(g_{ad,bc} + g_{bc,ad} - g_{ac,bd} - g_{bd,ac}\right) - \Gamma_{eac}\Gamma^e_{\ bd} + \Gamma_{ead}\Gamma^e_{\ bc}\,.$$

4. (Extra) We have assumed that it is always possible to make a coordinate transformation to a frame where all inertial and gravitational forces disappear locally. Now it is time to prove

[4] You might want to prove the following identities first,

$$\Gamma_{aed} = -\Gamma_{ead} + g_{ae,d}\,,$$

and

$$g_{ae}\Gamma^e_{bc,d} = \Gamma_{abc,d} - g_{ae,d}\Gamma^e_{bc}\,.$$

it formally. So, prove that it is always possible to choose a coordiante system in which all Christoffel symbols are zero at a given point.[5]

5. Prove that if at a given point in space the Christoffel symbols are zero, then the first (but not necessarily the second) derivatives of the metric tensor are also zero[6].

6. Prove the interchange symmetry of the Riemann tensor.

7. Prove the antisymmetry of the Riemann tensor.

8. Prove the algebraic Bianci identity.

9. Prove the differential Bianchi identity.

10. Prove that $(R^{ab} - \frac{1}{2}Rg^{ab})_{;a} = 0$.

11. Compute the Riemann tensor for the space with the metric $ds^2 = dr^2 + r^2 d\phi^2$ and discuss the result.

12. Compute all non-vanishing components of the Riemann tensor R_{abcd} (where each of indices a, b, c, d takes the values θ, ϕ) for the metric

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{35}$$

on a 2-dimensional sphere of radius r. Calculate also the Ricci tensor R_{ab} and the scalar curvature (Ricci scalar) R.

Answer:

$$R_{\theta\phi\theta\phi} = r^2\sin^2\theta = R_{\phi\theta\phi\theta} = -R_{\theta\phi\phi\theta} = -R_{\phi\theta\theta\phi}$$

[5]The transformation rule for the Christoffel symbols between a non-primed and a primed coordinate systems is given as

$$\Gamma^a_{bc} = \frac{\partial x^a}{\partial x'^e}\frac{\partial x'^f}{\partial x^b}\frac{\partial x'^d}{\partial x^c}\Gamma'^e_{fd} + \frac{\partial^2 x'^e}{\partial x^b \partial x^c}\frac{\partial x^a}{\partial x'^e} .$$

Consider, for simplisity, the origin and prove that the following coordinate transformation,

$$x'^a = x^a + \frac{1}{2}\left(\Gamma^a_{bc}\right)_0 x^b x^c ,$$

where $\left(\Gamma^a_{bc}\right)_0$ is the Christoffel symbol at the origin in the non-primed system, reduces the primed Christoffel symbols at the origin, $\left(\Gamma'^d_{ef}\right)_0$, to zero.

[6]Express the derivatives of the metric tensor as linear combinations of Christoffel symbols.