Electrodynamics in gravitational fields

In this section we shall use the principle of stationary action to derive the Lorentz force equation[1] and the Maxwell equations[2] in curvilinear coordinates. Due to the equivalence principle these equations should also be valid in the genuinely curved spaces of gravitational fields.

Principle of stationary action

In classical physics the equations of motion of a physical system can be conveniently derived using the *principle of stationary action*[3] (also called the *variational principle*, or the *least-action principle*):

> *The path actually followed by a physical system is that for which the action, S, is stationary, that is to say that its variation vanishes,*
>
> $$\delta S = 0\,. \tag{4}$$

Provided the action is written in a generally covariant form in curvilinear coordinates, the principle is applicable to non-inertial frames and—by the equivalence principle—also to gravitational fields.

Lorentz force equation

In special relativity the action of a body with mass m and charge e, moving in a given electromagnetic field A^a, is given (in the units $c = 1$) as

$$S = -m\int ds - e\int dx^a A_a\,, \tag{5}$$

where the integrals are taken along the trajectory of the body. In this form the action is generally covariant and can be directly used in general relativity. One only has to remember that the metric tensor is generally not constant throughout the space.

We shall calculate the variation of this action under a small variation of the trajectory of the body,

$$x^a \to x^a + \delta x^a\,. \tag{6}$$

The variation of the first term in (5) has already been calculated in the section about geodesics,

$$\delta\left(-m\int ds\right) = \int ds\delta x^a m\left(\frac{du_a}{ds} - \frac{1}{2}g_{bc,a}u^b u^c\right)\,. \tag{7}$$

[1]The *Lorentz force* equation is the equation of motion of a charged particle in a given electromagnetic field. In special relativity it is often written in 3D-notation as

$$\frac{d\vec{p}}{dt} = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B}\,, \tag{1}$$

where $\vec{p}$ is the relativistic momentum of the body, t is the time-coordinate, e is the charge of the particle, $\vec{E}$ is the electric field and $\vec{B}$ is the magnetic field, and the sign $\times$ denotes the vector-product.

[2]The *Maxwell equations* are the equation of motion of the electromagnetic field. In special relativity they are often written in 3D-notation as

$$\nabla\vec{B} = 0\,,\ \nabla\times\vec{E} + \frac{\partial\vec{E}}{c\partial t} = 0\,,\ \nabla\vec{E} = 4\pi\rho\,,\ \nabla\times\vec{B} - \frac{\partial\vec{E}}{c\partial t} = 4\pi\frac{\vec{j}}{c}\,, \tag{2}$$

where $\nabla \doteq \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$, and $\vec{j}$ is the 3D-current.

[3]*Action* is an attribute of the dynamics of a physical system: it is a functional that takes the trajectory of the system (also called *path* or *history*) as its argument and returns a (covariant) real scalar as the result. If the action is represented as an integral over time, taken along the path of the system between the initial time and the final time of the development of the system,

$$S = \int \mathcal{L}dt\,, \tag{3}$$

the integrand $\mathcal{L}$ is called the *Lagrangian*.

The variation of the term $dx^a A_a$ is given as[4]

$$\delta\left(-e\int dx^a A_a\right) = -e\int ds\delta x^a\left(-A_{a,b}+A_{b,a}\right)u^b\ , \tag{11}$$

where the expression in parentheses is called the *electromagnetic tensor*, F_{ab},

$$F_{ab} \doteq -A_{a,b} + A_{b,a}\ . \tag{12}$$

It is apparently antisymmetric, $F_{ab} = -F_{ba}$. Although it contains non-covariant derivatives it is actually a tensor since in a torsion free space it can be also written through covariant derivatives,

$$F_{ab} = -A_{a;b} + A_{b;a}\ . \tag{13}$$

The variation of the action (5) then becomes

$$\delta S = \int ds\delta x^a\left(m\frac{du_a}{ds} - m\frac{1}{2}g_{bc,a}u^b u^c - eF_{ab}u^b\right)\ . \tag{14}$$

Since the variation δx^a is arbitrary, $\delta S = 0$ means the expression in parentheses has to vanish identically on physical trajectories, giving the the equation of motion of a charged body in both gravitational and electro-magnetic fields (the generalization of the *Lorentz force* equation),

$$m\frac{du_a}{ds} - m\frac{1}{2}g_{bc,a}u^b u^c = eF_{ab}u^b\ . \tag{15}$$

It can also be written as

$$m\frac{du_a}{ds} - m\Gamma_{bca}u^b u^c = eF_{ab}u^b\ , \tag{16}$$

or as

$$m\frac{Du_a}{ds} = eF_{ab}u^b\ . \tag{17}$$

Maxwell equations in curvilinear coordinates

In this section we shall derive the generally covariant equations for the electromagnetic field – the Maxwell equations in curvilinear coordinates.

Homogeneous Maxwell equation

The generally covariant form of the homogeneous Maxwell equation can be deduced from its form in Minkowski space,

$$F_{ab,c} + F_{bc,a} + F_{ca,b} = 0\,, \tag{18}$$

where

$$F_{ab} \doteq -A_{a,b} + A_{b,a} \tag{19}$$

is the electromagnetic tensor. Actually, due to the symmetry of Christoffel symbols, $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$, and antisymmetry of the electromagnetic tensor, $F_{ba} = -F_{ab}$, both the electromagnetic tensor

[4]

$$\delta(dx^a A_a) = \delta dx^a A_a + u^a ds A_{a,b}\delta x^b = \delta dx^a A_a + u^b ds A_{b,a}\delta x^a\ . \tag{8}$$

As usual the term $\delta dx^a A_a$ is integrated by parts,

$$\delta dx^a A_a = d(\delta x^a A_a) - \delta x^a dA_a = d(\delta x^a A_a) - \delta x^a A_{a,b}u^b ds\,. \tag{9}$$

Inserting this into (8) leads to

$$\delta(dx^a A_a) = d(\delta x^a A_a) + (-A_{a,b} + A_{b,a})u^b ds\delta x^a\ . \tag{10}$$

The full differential in (10) as usual does not contribute to the variation since δx^a is zero at the end-points of the trajectory.

and the homogeneous Maxwell equation are already generally covariant and thus preserve their form in curvilinear coordinates. Indeed,

$$F_{ab} = -A_{a,b} + A_{b,a} = -A_{a;b} + A_{b;a}\,, \tag{20}$$

where the latter expression is generally covariant. Then

$$F_{ab,c} + F_{bc,a} + F_{ca,b} = F_{ab;c} + F_{bc;a} + F_{ca;b}\,, \tag{21}$$

which is also generally covariant.

Inhomogeneous Maxwell equation

For the inhomogeneous Maxwell equation we shall first deduce the generally covariant action of the electromagnetic field, and then derive the equation from the stationary action principle.

In Minkowski space the action of the electromagnetic field is given as an integral over the whole space-time,

$$S = \int d\Omega \left(-\frac{1}{16\pi} F^{ab} F_{ab} - A_a j^a \right) . \tag{22}$$

Covariant volume element

The infinitesimal volume element,

$$d\Omega \doteq d^4 x\,, \tag{23}$$

in this action is not invariant under a general coordinate transformation and has to be substituted with the generally covariant volume element,

$$d\Omega \to \sqrt{-g}\, d\Omega\,, \tag{24}$$

where g is the determinant of the metric tensor g_{ab} ($g < 0$).

Indeed, the metric tensor transforms as

$$g_{ab} = \frac{\partial x'^c}{\partial x^a} \frac{\partial x'^d}{\partial x^b} g'_{cd}\,. \tag{25}$$

Taking determinant of both sides gives $g = J'^2 g'$, or

$$\sqrt{-g} = J' \sqrt{-g'}\,, \tag{26}$$

where $J' = \left| \frac{\partial x'^a}{\partial x^b} \right|$ is the Jacobian determinant of the transformation. The 4-volume transforms as

$$d\Omega = \left| \frac{\partial x^a}{\partial x'^b} \right| d\Omega' = \frac{1}{J'} d\Omega'. \tag{27}$$

Apparently the combination $\sqrt{-g}\, d\Omega$ transforms as

$$\sqrt{-g}\, d\Omega = J' \sqrt{-g'} \frac{1}{J'} d\Omega' = \sqrt{-g'}\, d\Omega', \tag{28}$$

and is thus generally invariant.

The generally covariant action of the electromagnetic field is thus given as

$$S = \int \sqrt{-g}\, d\Omega \left(-\frac{1}{16\pi} F^{ab} F_{ab} - A_a j^a \right) . \tag{29}$$

Variation of the action and the inhomogeneous Maxwell equation

The variation of the electromagnetic field, $A_a \to A_a + \delta A_a$, in the second term of the action produces

$$-\delta \int \sqrt{-g} d\Omega A_a j^a = -\int d\Omega \sqrt{-g} j^a \delta A_a \,. \tag{30}$$

The first term gives

$$-\frac{1}{16\pi}\delta \int \sqrt{-g} d\Omega F^{ab} F_{ab} = +\frac{1}{4\pi}\int d\Omega \sqrt{-g} F^{ab} (\delta A_a)_{,b} = -\frac{1}{4\pi}\int d\Omega \left(\sqrt{-g} F^{ab}\right)_{,b} \delta A_a \tag{31}$$

Combining the two terms gives the variation of the electromagnetic action,

$$\delta S = \int d\Omega \delta A_a \left(-\frac{1}{4\pi}\left(\sqrt{-g} F^{ab}\right)_{,b} - \sqrt{-g} j^a \right) , \tag{32}$$

from which directly follows the sought inhomogeneous Maxwell equation in curvilinear coordinates,

$$\left(\sqrt{-g} F^{ab}\right)_{,a} = 4\pi \sqrt{-g} j^b \,. \tag{33}$$

It can also be written in an explicitly covariant form,

$$F^{ab}_{;a} = 4\pi j^b \,, \tag{34}$$

using the identity

$$\sqrt{-g} V^a_{;a} = (\sqrt{-g} V^a)_{,a} \,. \tag{35}$$

To prove that, we shall first need the formula for the variation δg of the determinant g of the metric tensor (the Jacobi's formula). Suppose we vary one element, say g_{23}, of the metric tensor. The contribution of this element to the determinant of the metric tensor, g, is given as

$$g_{23} C_{23} = g_{23} g g^{23} \,, \tag{36}$$

where C_{23} is the cofactor of the element g_{23}, and g^{23} is the element of the inverse metric tensor. Then, apparently[5],

$$\delta g = g \delta g_{ab} g^{ab} = -g \delta g^{ab} g_{ab} \,. \tag{37}$$

Now let us consider the covariant divergence of the electromagnetic tensor,

$$F^{ab}_{;a} = F^{ab}_{,a} + \Gamma^a{}_{da} F^{db} \,. \tag{38}$$

The contraction $\Gamma^a{}_{da}$ of the Christoffel symbol is given as

$$\Gamma^a{}_{da} = \frac{1}{2} g^{ab} g_{ab,d} = \frac{1}{2g} g_{,d} \,. \tag{39}$$

The divergence then becomes

$$F^{ab}_{;a} = F^{ab}_{,a} + \frac{1}{2g} g_{,d} F^{db} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g} F^{ab}\right)_{,a} , \tag{40}$$

which concludes the proof.

[5] In matrix calculus this formula is called the Jacobi's formula.

Exercises

1. Argue that
$$\frac{du_a}{ds} - \frac{1}{2}g_{bc,a}u^b u^c = \frac{du_a}{ds} - \Gamma_{bca}u^b u^c = \frac{Du_a}{ds} \ . \tag{41}$$

2. Consider the following action of a body,
$$S = \int ds \mathcal{L}(x^a, u^a) \ ,$$
where ds is the invariant interval, $u^a \doteq dx^a/ds$, and where the intergal is taken along the path of the body (and where the function $\mathcal{L}$ is called the Lagrangian). Argue that the corresponding Euler-Lagrange equation is given as
$$\frac{d}{ds}\frac{\partial \mathcal{L}}{\partial u^c} = \frac{\partial \mathcal{L}}{\partial x^c} \ .$$
Using this equation, derive the Lorentz force equation in general relativity from the action
$$S = -m\int ds - e\int dx^a A_a = \int ds \left(-m\sqrt{g_{ab}u^a u^b} - eu^a A_a\right) \ .$$

3. Consider the following action of a scalar field ϕ,
$$S = \int d\Omega \mathcal{L}(\phi, \phi_{,a}) \ ,$$
where $\phi_{,a} \doteq \partial\phi/\partial x^a$. Argue that the corresponding Euler-Lagrange equation is given as
$$\frac{\partial}{\partial x^c}\frac{\partial \mathcal{L}}{\partial \phi_{,c}} = \frac{\partial \mathcal{L}}{\partial \phi} \ .$$

4. Consider the following action of a vector field A_a,
$$S = \int d\Omega \mathcal{L}\left(A_a, A_{a,b}\right) \ .$$
Argue that the corresponding Euler-Lagrange equation is given as
$$\frac{\partial}{\partial x^c}\frac{\partial \mathcal{L}}{\partial A_{d,c}} = \frac{\partial \mathcal{L}}{\partial A_d} \ .$$
Using this equation, derive the Maxwell equations (with sources) in general relativity from the action
$$S = \int d\Omega \left(-\frac{1}{16\pi}\sqrt{-g}F^{ab}F_{ab} - \sqrt{-g}A_a j^a\right) \ .$$

5. (Extra) In the Minkowski space of special relativity,
   (a) Derive the Lorentz force equation,
   $$m\frac{du_a}{ds} = eF_{ab}u^b \ ,$$
   from the action
   $$S = -m\int ds - e\int dx^a A_a \ .$$

(b) Show that the Lorentz force equation can be written in 3D-notation as

$$\frac{d\vec{p}}{dt} = e\vec{E} + \frac{e}{c}\vec{v} \times \vec{B} \,,$$

where $\vec{p}$ is the relativistic momentum and where

$$A^a = \{\phi, \vec{A}\} \,,\; \vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \,,\; \vec{B} = \nabla \times \vec{A} \,.$$

6. (Extra) In the Minkowski space of special relativity,

   (a) from the action

   $$S = -\frac{1}{8\pi}\int d^4x A_{a,b}A^{a,b} - \int d^4x A^a j_a$$

   derive[6] the Maxwell equations with sources,

   $$A^{b,a}_{,a} = 4\pi j^b \,.$$

   (b) Show that with the Lorenz condition,

   $$A^a_{,a} = 0 \,,$$

   they are equivalent to

   $$F^{ab}_{,a} = 4\pi j^b \,.$$

   (c) Rewrite the latter using 3D-notation.

7. Argue that the electromagnetic tensor F_{ab} is a generally covariant tensor by proving that

$$F_{ab} \doteq -A_{a,b} + A_{b,a} = -A_{a;b} + A_{b;a} \,. \tag{42}$$

8. (a) Argue that in Minkowski space the electromagnetic tensor F_{ab} satisfies the homogeneous Maxwell equation,

   $$F_{ab,c} + F_{bc,a} + F_{ca,b} = 0 \,.$$

   (b) Rewrite the latter in 3D-notation.

   (c) Argue that in curvilinear coordinates the electromagnetic tensor satisfies the generally covariant form of this equation,

   $$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0 \,.$$

9. (a) Derive (by direct variation) the (inhomogeneous) Maxwell equation in curvilinear coordinates,

   $$\left(\sqrt{-g}F^{ab}\right)_{,a} = 4\pi\sqrt{-g}j^b \,,$$

   from the action

   $$S = \int \left(-\frac{1}{16\pi}F^{ab}F_{ab} - A_a j^a\right)\sqrt{-g}d\Omega \,.$$

[6]For fields the usual "integration by parts" is done using the Gauss theorem in Minkowski space,

$$\int_\Omega d^4x \frac{\partial B^a}{\partial x^a} = \oint_{\partial\Omega} B^a dS_a \,,$$

where dS_a is an infinitesimal element of the hyper-surface $\partial\Omega$ of the 4-volume Ω. In Euclidean 3D space it has a more familiar form,

$$\int_V dV\left(\vec{\nabla}\cdot\vec{B}\right) = \int_{\partial V} dS\left(\vec{B}\cdot\vec{n}\right) .$$

(b) Show that the equation can also be written as

$$F^{ab}_{;a} = 4\pi j^b \,.$$

Hints:

i. show that $\Gamma^a{}_{ba} = \frac{1}{2g} g_{,b} = (\ln\sqrt{-g})_{,b}$

ii. show that $F^{ab}_{;a} = \frac{1}{\sqrt{-g}}(\sqrt{-g}F^{ab})_{,a}$

(c) Show that from this equation it follows, that

$$\left(\sqrt{-g}j^a\right)_{,a} = 0 = j^a_{;a} \,.$$