# Motion of free bodies in gravitational fields

General relativity (like the second Nordström's theory) is a geometric theory: the gravitational field modifies the metric tensor of the space-time, making the space-time curved. In geometric theories free bodies (that is, bodies not affected by electromagnetic or other matter[1] forces) move along curves called *geodesics* which are the curved-space analogues of the of flat-space *lines*. Massive bodies distort space-time in their vicinity causing the geodesics to bend.

## Geodesics

Einstein's principle of equivalence implies that a law of physics of special relativity formulated in a generally covariant form should also be valid in general relativity. Therefore the law of motion of free bodies in gravitational fields can be conveniently obtained by a suitable generalization of the corresponding law of special relativity.

### Geodesic as "constant" velocity trajectory

In special relativity (in an inertial frame with cartesian coordinates) a free body moves with constant velocity, that is, the differential of the velocity along the body's path is zero,

$$du^a = 0\,, \tag{1}$$

where the velocity vector $u^a$ of a moving body is defined as

$$u^a = \frac{dx^a}{ds}\,, \tag{2}$$

where $dx^a$ is the infinitesimal movement of the body along the trajectory and $ds = \sqrt{g_{ab}dx^a dx^b}$ is the invariant interval (the metric).

The velocity $u^a$ is a vector (that is, a covariant quantity) and can be directly used to write equations in curvilinear coordinates and/or curved spaces. However the differential $du^a$ is not covariant and therefore equation (1) is not covariant and cannot be used in the curved spaces of general relativity.

A suitable generalization to curved spaces would be to substitute the ordinary differential $du^a$ with the covariant differential $Du^a$,

$$Du^a = 0\,. \tag{3}$$

This equation is generally covariant and is actually the *geodesic equation*. It can be also written as

$$\frac{du^a}{ds} + \Gamma^a{}_{bc}u^b u^c = 0\,, \tag{4}$$

or

$$\frac{d^2x^a}{ds^2} + \Gamma^a{}_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0\,. \tag{5}$$

One can interpret this equation as the relativistic generalization of Newton's second law of motion: acceleration of the body equals the force acting on the body (divided by mass). Therefore the quantities $(-)\Gamma^a{}_{bc}u^b u^c$ appear as forces—inertial and gravitational—acting on the body. Since the Christoffel symbols (and hence the forces) are proportional to derivatives of the metric tensor, one can say that the metric tensor plays the role of the potential of gravitational forces, like in Nordström's theory. However unlike the Nordström's theory in general relativity there is no *conformal* restrictions on the metric tensor. The gravitational field in general relativity has all the degrees of freedom of a (symmetric) tensor and is thus a tensor field.

[1] In general relativity "matter" is "everything else but gravity and inertia".

### Geodesic as extremal trajectory

Equivalently the trajectories of free bodies in special relativity can be postulated to be the curves with extremal measure[2] (which in special relativity happen to be lines, that is, straight curves). This postulate is generally covariant (indeed, the measure of a curve is a quantity that is invariant under a general coordinate transformation) and can be directly applied to curvilinear coordinates and/or gravitational fields. Therefore in general relativity free bodies[3] are also postulated to move along curves with extremal measures.

A curve with extremal measure is called a *geodesic*. Geodesic is a generalization of the notion of a *line* to curved spaces. The term comes from *geodesy*, the science of measuring the size and shape of Earth. In the original sense, a geodesic was the shortest route between two points on the Earth's surface, namely a segment of a great circle. The term has since been generalized to include measurements in more general mathematical spaces.

Importantly, the postulate of free motion along geodesics can be formulated as a variational (least action) principle: the variation of the measure vanishes on the curve actually taken by the free body,

$$\delta \int ds = 0 \,. \tag{6}$$

The measure $\mu$ of a trajectory (of a moving body) is defined as the sum of infinitesimal intervals $ds$ along the trajectory,

$$\mu = \int ds \,. \tag{7}$$

The extremal trajectory is the one where the variation of the measure as function of the trajectory vanishes,

$$\delta\mu = 0 \,. \tag{8}$$

To calculate the variation of the measure we first vary the interval $ds$,

$$\delta ds = \delta\sqrt{g_{ab}dx^a dx^b} = \frac{1}{2}\frac{1}{\sqrt{g_{ab}dx^a dx^b}}\delta\left(g_{ab}dx^a dx^b\right) = \frac{1}{2}\frac{1}{ds}\left(\delta g_{ab}dx^a dx^b + 2g_{ab}\delta dx^a dx^b\right) \,. \tag{9}$$

Using $\delta g_{ab} = g_{ab,c}\delta x^c$ and $\frac{1}{ds}g_{ab}dx^b = u_a$ gives[4]

$$\delta ds = \frac{1}{2}g_{ab,c}u^a u^b \delta x^c ds + \delta dx^c u_c \,, \tag{10}$$

where in the second term the summation index $a$ was replaced with $c$.

Assuming the functions are smooth enough we can exchange the order of differentials in the second term and integrate it by parts using

$$\delta dx^c u_c = d\delta x^a u_a = d(\delta x^a u_a) - \delta x^c du_c. \tag{11}$$

The full differential does not contribute to the variation, and we finally arrive at

$$\delta\mu = \int ds \delta x^c \left(-\frac{du_c}{ds} + \frac{1}{2}g_{ab,c}u^a u^b\right) = 0 \,. \tag{12}$$

Since the variation $\delta x$ is arbitrary, it is the expression in parentheses that should be equal zero identically along the trajectory, which gives the following equation for the curve with extremal measure,

$$\frac{du_c}{ds} = \frac{1}{2}g_{ab,c}u^a u^b \,. \tag{13}$$

[2] The measure of a curve is given by the integral $\int ds$ (where $ds$ is the metric) taken along the curve.

[3] In general relativity a free body is a body that is free from the influence of *matter forces*; gravitational and intertial forces do not count as matter.

[4]
$$f_{,a} \doteq \frac{\partial f}{\partial x^a} \,.$$

This equation is equivalent to the no-acceleration equation (3)[5]. Thus the trajectory with extremal measure is also the trajectory along which the covariant differential of the velocity of freely moving body is zero. This is the consequence of the fact that the action $S$ of a free body with mass $m$ is proportional to the measure of its trajectory,

$$S = -mc \int ds \,. \tag{17}$$

## Trajectories of rays of light

The equation (3) is not applicable to the propagation of a ray of light since the interval $ds$ along the ray is always zero. In this case one has to use some parameter, $\lambda$, which varies smoothly along the ray. Then one can introduce the wave-vector, $k^a = dx^a/d\lambda$. In special relativity the ray of light propagates along a line where $dk^a = 0$. In a curved space in analogy with (3) this becomes

$$Dk^a = 0 \,, \tag{18}$$

or

$$\frac{dk^a}{d\lambda} + \Gamma^a{}_{bc} k^b k^c = 0 \,. \tag{19}$$

These equations, together with the condition that for the ray of light always $k^a k_a \propto dx^a dx_a = ds^2 = 0$, also determine the parameter $\lambda$ should one wish to find it.

The trajectories of the rays of light are called *null geodesics*.

## Exercises

1. Consider a generalization of the Galilean (non-relativistic) Lagrangian of a free body in a Euclidean space,
$$\mathcal{L} = \frac{1}{2} m \mathbf{v}^2 \,,$$
to curvilinear coordinates with the metric $dl^2 = g_{\alpha\beta} x^\alpha x^\beta$ (where $\alpha, \beta = 1, 2, 3$),
$$\mathcal{L} = \frac{1}{2} m \mathbf{v}^2 \;\rightarrow\; \mathcal{L} = \frac{1}{2} m g_{\alpha\beta} v^\alpha v^\beta \,,$$
where $v^\alpha \doteq dx^\alpha/dt$. Show that the corresponding Euler-Lagrange equation[6] is the geodesic equation for the given metric.

[5]Indeed, from

$$0 = Dg_{ab} = dg_{ab} - \Gamma^e{}_{ac} g_{eb} dx^c - \Gamma^e{}_{bc} g_{ae} dx^c = (g_{ab,c} - \Gamma_{bac} - \Gamma_{abc}) dx^c \tag{14}$$

it follows that

$$g_{ab,c} = \Gamma_{bac} + \Gamma_{abc} \,. \tag{15}$$

Inserting this into (13) and using the symmetry $u^a u^b = u^b u^a$ gives

$$\frac{du_c}{ds} - \frac{1}{2} g_{ab,c} u^a u^b = \frac{du_c}{ds} - \frac{1}{2}\left(\Gamma_{bac} u^a u^b + \Gamma_{abc} u^a u^b\right) = \frac{du_c}{ds} - \Gamma_{abc} u^a u^b = \frac{Du_c}{ds} \,, \tag{16}$$

which had to be demonstrated.

[6]The Euler-Lagrange equation for the action $S = \int dt \mathcal{L}(x^\mu, v^\nu)$ where $v^\nu = dx^\nu/dt$, is given as

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial v^\mu} = \frac{\partial \mathcal{L}}{\partial x^\mu} \,,$$

or, for the action $S = \int ds \mathcal{L}(x^a, u^b)$ where $u^b = dx^b/ds$, as

$$\frac{d}{ds} \frac{\partial \mathcal{L}}{\partial u^a} = \frac{\partial \mathcal{L}}{\partial x^a} \,.$$

2. Consider a further generalization to a relativistic action,

$$S = \int \frac{1}{2} m g_{\alpha\beta} v^\alpha v^\beta dt \;\rightarrow\; S = -m \int g_{ab} u^a u^b ds \;,$$

where $u^a \doteq dx^a/ds$ and $a, b = 0, 1, 2, 3$. Show that the corresponding Euler-Lagrange equation is the relativistic geodesic equation.

3. Show that the Euler-Lagrange equation for the action

$$S = -m \int ds = \int \left( -m\sqrt{g_{ab} dx^a dx^b} \right) = \int \left( -m\sqrt{g_{ab} \frac{dx^a}{ds} \frac{dx^b}{ds}} \right) ds \mathrel{\hat{=}} \int \mathcal{L}\left( x^a, \frac{dx^a}{ds} \right) ds \;,$$

is the relativistic geodesic equation.

4. (Extra) Show that the Euler-Lagrange equation for the action

$$S = -m \int ds f(g_{ab} u^a u^b) \,,$$

where $f$ is a differential function of one argument, is the geodesic equation.

5. In a scalar theory of gravitation consider the following action of a body with mass $m$ moving in a scalar gravitational field $\Phi$,

$$S = -mc \int \sqrt{\eta_{ab} dx^a dx^b} - mc \int \Phi \sqrt{\eta_{ab} dx^a dx^b} \;. \tag{20}$$

Show the the corresponding Euler-Lagrange equation is equivalent to geodesic equation for the metric tensor $g_{ab} = (1 + \Phi)^2 \eta_{ab}$.

6. Prove that the two postulates,

$$D u_c = 0 \;,$$

and

$$\delta \int ds = 0 \,,$$

for the motion of free bodies in curved spaces are equivalent[7].

7. Consider the parametric equations for a line in Cartesian coordinates $x$ and $y$,

$$\frac{d^2 x}{d\lambda^2} = 0 \,,\; \frac{d^2 y}{d\lambda^2} = 0 \;, \tag{21}$$

where $\lambda$ is a parameter that smoothly changes along the curve. Make a coordinate transformation to polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$) and derive the corresponding equations in the $r, \theta$ coordinates. Prove that they are identical to the geodesic equation (5).

8. Apply the generally covariant equation for the trajectory of a ray of light, $Dk^a/d\lambda = 0$ (where $k^a = dx^a/d\lambda$, where $x^a(\lambda)$ follows the trajectory), to a flat two-dimensional space with polar coordinates and find the corresponding trajectories[8].

[7] That is, pove that the equations of motion (4) and (13) are equivalent.

[8] The first of the two geodesic equations,

$$\frac{d}{d\lambda}\left( r^2 \frac{d\phi}{d\lambda} \right) = 0 \,,\; \text{and} \; \frac{d^2 r}{d\lambda^2} = r \left( \frac{d\phi}{d\lambda} \right)^2 \;,$$

is easily integrated,

$$r^2 \frac{d\phi}{d\lambda} = J \,,$$

where $J$ is the integration constant. The second is traditionally integrated by a substitution, $r(\lambda) = 1/u(\phi(\lambda))$, which transforms the second equation into

$$\frac{d^2 u}{d\phi^2} + u = 0$$

with the general solution $u = u_0 \sin(\phi - \phi_0)$ which is a general parametrization of a line in polar coordinates.

9. (extra) Find the equations for geodesics on the surface of a sphere. Hint:[9] integrate the $\phi$-equation once and then rewrite the $\theta$-equation for the function $u(\phi) = \cot(\theta(\phi))$.

[9]The geodesic equations are

$$\ddot{\theta} = \sin\theta\cos\theta\dot{\phi}^2 \ , \quad \frac{d}{dt}\left(\sin^2\theta\dot{\phi}\right) = 0 \ ,$$

where dot denotes time-derivative, $\dot{\phi} \,\hat{=}\, d\phi/dt$. The second equation can be easily integrated,

$$\sin^2\theta\dot{\phi} = k \ ,$$

where $k$ is the integration constant. The first equation must be rewritten for the function $\theta(\phi(t))$ instead of $\theta(t)$. The second $t$-derivative of $\theta$ then becomes (using $\dot{\phi} = k/\sin^2\theta$),

$$\ddot{\theta} = -\frac{k^2}{\sin^2\theta}\left(-\frac{\theta''}{\sin^2\theta} + 2\frac{\cos\theta}{\sin^3\theta}\theta'^2\right) = -\frac{k^2}{\sin^2\theta}\left(\cot\theta\right)'' \ ,$$

where ptime denotes $\phi$-derivative, $\theta' \,\hat{=}\, d\theta/d\phi$. The first geodesic equation then becomes (again using $\dot{\phi} = k/\sin^2\theta$)

$$-(\cot\theta)'' = \cot\theta \ ,$$

with the general solution

$$\cot\theta = A\sin(\phi - \phi_0) \ ,$$

(where $A$ and $\phi_0$ are integration constats) which is the equation for a great circle.