Proper time and length intervals

Proper time interval

Just like in special relativity the proper time interval $d\tau$ is given by the interval between two close events with the same spatial coordinates,

$$d\tau^2 \doteq ds^2\Big|_{dx^\alpha=0} = g_{00}dt^2 \ . \tag{1}$$

The connection between the proper time $d\tau$ and the coordinate time dt is then given as

$$d\tau = \sqrt{g_{00}}dt \ . \tag{2}$$

Proper length interval

In general relativity we cannot define the proper length interval dl by putting $dt = 0$ in ds. Indeed time runs differently in different points.

We can define the proper length from one pont to another point as in the following: we send a light signal from the point to another point; another point reflects the light and sends it back to the first point; the proper length between the two points is then defined as half the proper time between sending the light signal from the first point and receiving it back.

Let us send a light signal between two points separated by dx^a. For a light signal $ds^2 = 0$ therefore

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + 2g_{0\alpha}dtdx^\alpha + g_{00}dt^2 = 0 \ . \tag{3}$$

Solving this for dt gives

$$dt^{(\text{there})} = \frac{-2g_{0\alpha}dx^\alpha + \sqrt{4(g_{0\alpha}g_{0\beta} - g_{00}g_{\alpha\beta})dx^\alpha dx^\beta}}{2g_{00}} \ . \tag{4}$$

For the light signal sent in the opposite direction, $-dx^\alpha$,

$$dt^{(\text{back})} = \frac{+2g_{0\alpha}dx^\alpha + \sqrt{4(g_{0\alpha}g_{0\beta} - g_{00}g_{\alpha\beta})dx^\alpha dx^\beta}}{2g_{00}} \ . \tag{5}$$

The total time,

$$dt = dt^{(\text{there})} + dt^{(\text{back})} = 2\frac{\sqrt{(g_{0\alpha}g_{0\beta} - g_{00}g_{\alpha\beta})dx^\alpha dx^\beta}}{g_{00}} \ . \tag{6}$$

Now, $\sqrt{g_{00}}dt$ is the proper time between sending the signal and receiving it at the same point. The proper length is then $\sqrt{g_{00}}dt/2$,

$$dl^2 = \left(-g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}\right)dx^\alpha dx^\beta \ . \tag{7}$$