Big Bang

The Friedman equation can be integrated analytically for the important cases of mass-dominated universe (the current universe) and radiation-dominated universe (the early universe). The solutions suggest that a Friedman universe starts its existence from a state with extremely high energy density, followed by a period of expansion. This scenario is called the *Big Bang*.

Depending on the geometry, the universe either expands forever (for open and flat geometries) or ends its existence in a Big Crunch (for the closed geometry).

The expansion of the universe leads to cosmological red-shift – an apparent recessional velocity of distant galaxies proportional to their distances from Earth. Cosmological red-shift has been observed experimentally as the Hubble's law.

Mass-dominated universe

Mass-dominated universe has zero pressure, $p = 0$, and its energy density equal its mass density, $\epsilon = \mu$. The energy conservation equation,

$$d\epsilon = -(\epsilon + p)\frac{3da}{a}\,, \tag{1}$$

turns into the mass conservation equation,

$$d\mu = -\mu\frac{3da}{a}\,, \tag{2}$$

with the solution

$$\mu a^3 = \mathrm{const}_1\,. \tag{3}$$

With this energy density the Friedman equation for a closed universe,

$$d\eta = \pm\frac{da}{a\sqrt{\frac{1}{3}\kappa\epsilon a^2 - 1}}\,, \tag{4}$$

gives, taking the positive square root[1],

$$d\eta = \frac{da}{\sqrt{2a_0 a - a^2}} = d\arccos\frac{a_0 - a}{a_0}\,, \tag{5}$$

where $a_0 = \frac{1}{6}\kappa\,\mathrm{const}_1$.

Integrating first (5) and then $dt = ad\eta$ gives the solution for the closed universe,

$$a = a_0(1 - \cos(\eta))\,,\ t = a_0(\eta - \sin(\eta)) \tag{6}$$

The life time of a closed universe is finite, $\Delta\eta = 2\pi$, the universe starts with the Big Bang at $\eta = 0$, where $a \to 0$ and $\epsilon \to \infty$, end ends with the Big Crunch at $\eta = 2\pi$ where again $a \to 0$ and $\epsilon \to \infty$.

For the open isotropic universe the Friedman equation reads

$$d\eta = \pm\frac{da}{a\sqrt{\frac{1}{3}\kappa\epsilon a^2 + 1}}\,. \tag{7}$$

A similar integration[2] for the matter-dominated universe with the equation of state $p = 0$ gives

$$a = a_0(\cosh(\eta) - 1)\,,\ t = a_0(\sinh(\eta) - \eta) \tag{8}$$

For the open universe the scenario is big-bang → expansion-forever.

For a flat isotropic universe,

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)\,, \tag{9}$$

the scenario is also big-bang → expansion-forever with $a \propto t^{2/3}$.

[1] the ± sign simply reflects the symmetry of the equation under the substitution $\eta \to -\eta$.

[2] or a substitution $\chi \to i\chi$, $\eta \to i\eta$, $a \to ia$ in (6).

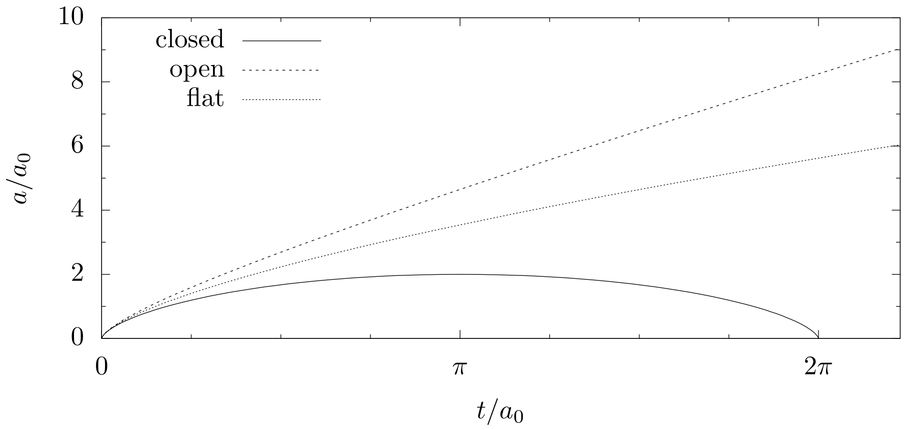

Figure 1: The scale $a(t)$ of a mass-dominated Friedman universe as function of time t for the open, closed, and flat geometries.

Radiation-dominated universe

At early stages with high densities the universe was (probably) rather radiation dominated, that is, filled with (noninteracting) photons.

The number of noninteracting photons in the universe is constant, which for the photon density, n, gives $na^3 = \text{const}$. However, the energy of a photon, $\hbar\omega$, scales with the size of the universe as a^{-1}. Therefore $\epsilon \propto na^{-1}$ and the energy conservation law becomes

$$\epsilon a^4 = \text{const}\,. \tag{10}$$

The increased pressure, however, does not save the universe from the singularity at "the beginning". Indeed, integrating the Friedman equation with $\epsilon a^4 = \text{const}$ for early times, $\eta \ll 1$, gives

$$a \propto t^{1/2}\,. \tag{11}$$

Cosmological red-shift and Hubble constant

In an isotropic universe the radial ($d\theta = d\phi = 0$) propagation of light is described by

$$0 = ds^2 = a^2(d\eta^2 - d\chi^2)\,, \tag{12}$$

with the solution

$$\chi = \pm\eta + \text{const}\,. \tag{13}$$

Suppose two flashes of light are travelling radially in rapid succession one after another. Their temporal and spatial separations $\Delta\eta = \Delta\chi$ remain constant along their trajectory in the $\{\eta, \chi\}$ coordinates. However, in $\{t, r\}$ the the corresponding separations $\Delta t = a\Delta\eta$ and $\Delta r = a\Delta\chi$ vary with a such that along the light ray $a\omega$ and λ/a remain constant.

Therefore a ray of light with frequency ω_0 emitted at a distance χ and observed at the origin ($\chi = 0$) at time η has the frequency

$$\omega = \omega_0 \frac{a(\eta - \chi)}{a(\eta)} \approx \omega_0 (1 - \chi \frac{a'}{a})\,, \tag{14}$$

that is, red-shifted, if the universe expands ($a' > 0$).

The proper distance l to the source of light is $l = \chi a$. Thus the frequency shift z can be written as

$$z \equiv \frac{\omega_0 - \omega}{\omega_0} = \frac{a'}{a^2} l \equiv Hl, \tag{15}$$

where H is the so called Hubble constant,

$$H = \frac{a'}{a^2} = \frac{1}{a}\frac{da}{dt} \,. \tag{16}$$

The current empirical value of the Hubble constant is

$$H \approx \frac{1}{14.4\,\text{billion years}} \,. \tag{17}$$

Geometry of the universe, the Hubble constant, and the critical density

Inserting $\frac{a'}{a^2} = H$ into Friedman equation gives, for a closed universe:

$$\frac{\kappa\mu}{3} - H^2 = \frac{1}{a^2} > 0 \,, \tag{18}$$

for an open universe:

$$\frac{\kappa\mu}{3} - H^2 = -\frac{1}{a^2} < 0 \,, \tag{19}$$

and for a flat universe:

$$\frac{\kappa\mu}{3} - H^2 = 0 \,. \tag{20}$$

The density μ_c,

$$\frac{\kappa\mu_c}{3} = H^2 \,, \tag{21}$$

for which the universe is flat, is called the *critical density*.

The current measurement of the geometry of the Universe (e.g. by studying specific fluctuations — the so called "spots" — in the cosmic background radiation) indicate that the Universe is close to be flat, or that the relative density $\Omega = \mu/\mu_c$ is close to one (with an error about few per cent). This is often referred to as flatness problem.

The visible matter constitutes only about 5% of the critical density. A much greater contribution — about 23% — comes from the yet unidentified dark matter. The rest — about 72% — is attributed to *dark energy*.

The critical density is estimated to be about 5 protons per cubic meter, whereas the average density of barionic matter in the Universe is believed to be about 0.2 protons per cubic meter.

Cosmological constant

Einstein proposed to include an extra term, Λ, in the Einstein equation,

$$R_{ab} - \frac{1}{2}Rg_{ab} - \Lambda g_{ab} = \kappa T_{ab} \,, \tag{22}$$

originally in order to achieve a stationary, unchanging universe. Einstein abandoned the idea after the observation of the Hubble's law which indicated that the universe might not be stationary. However recent discoveries in observational cosmology renewed the interest in the cosmological constant.

Dark eneregy dominated universe

Cosmoligial constant is one possible realisation of the dark energy (the form of energy that accelerates the expansion of the universe). Indeed in a flat dark-energy dominated universe the Friedman equation reads

$$3\frac{\dot{a}^2}{a^2} = \Lambda \,. \tag{23}$$

The solution for the expanding universe is

$$a(t) \propto \exp\left(\sqrt{\frac{\Lambda}{3}}t\right) \,, \tag{24}$$

where the expansion is accelerating.

Exercises

1. Interpret the cosmological red-shift $\frac{\omega_0-\omega}{\omega_0} = Hl$ (l is the distance to the red-shifted galaxy) as a Doppler effect and calculate the velocity with which a galaxy appears to be moving relative to the observer.

2. Show that $\epsilon a^4 = \text{const}$ corresponds to the equation of state $p = \epsilon/3$.