Friedmann universe

The *Friedmann universe* is a cosmological model where the universe is assumed to be homogeneous, isotropic, and filled with *perfect fluid*.[1] The assumption of homogeneity and isotropicity of the universe is often referred to as the *cosmological principle*. Empirically, it seems to by justified on scales larger that 250 million light years.

The isotropic and homogeneous model is sometimes called the *Standard Model* of the present-day cosmology.

The *Friedmann equations* are the Einstein equations applied to the Friedmann universe as a whole. They describe the temporal evolution of the Friedmann universe. The solutions to these equations are generally called *Friedmann–Lemaître–Robertson–Walker metric*.

Metrics of spaces with constant curvature

A homogeneous and isotropic universe constitutes a *space with constant curvature*. Let us first look at two-dimensional spaces of constant curvature: a three-dimensional sphere (positive curvature), a pseudo-sphere (negative curvature), and a plane (zero curvature).

On a sphere the length element in the ordinary spherical coordinates is given as

$$dl^2 = a^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \,, \tag{1}$$

where a is the radius of the sphere.

Let us introduce the polar coordinates $\{r, \phi\}$ on the sphere, with r measuring the distance to the north pole. The length of a circle around north pole, $\theta = \text{const}$, is equal $2\pi a \sin\theta$. Therefore if we want the circumference of the circle to be equal $2\pi r$ (and thus r to be the *reduced-circumference radius*) we need to define $r = a\sin\theta$. The length element in our polar coordinates becomes

$$dl^2 = \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2 d\phi^2 \,. \tag{2}$$

On a pseudo-sphere, where the curvature is negative, the length element is correspondingly

$$dl^2 = \frac{dr^2}{1 + \frac{r^2}{a^2}} + r^2 d\phi^2 \,. \tag{3}$$

In angular coordinates, where $r = a\sinh\theta$, the length element becomes

$$dl^2 = a^2 \left(d\theta^2 + \sinh^2\theta d\phi^2 \right) \,. \tag{4}$$

On a plane, where the curvature is zero, the length element is

$$dl^2 = dr^2 + r^2 d\phi^2 \,. \tag{5}$$

Analogously, a three-dimensional space with constant curvature can have one of the following three possible geometries:

• *flat* (zero curvature),

$$dl^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) = a^2 \left(d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2) \right) \,, \tag{6}$$

where $r = a\chi$, $\chi \in [0, \infty]$;

• *closed* (positive curvature),

$$dl^2 = \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) = a^2 \left(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2) \right) \,, \tag{7}$$

[1] Perfect fluids have no shear stresses, no viscosity, and no heat conduction.

where $r = a\sin\chi$, $\chi \in [0,\pi]$;

- and *open* (negative curvature),

$$dl^2 = \frac{dr^2}{1+\frac{r^2}{a^2}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) = a^2\left(d\chi^2 + \sinh^2\chi(d\theta^2+\sin^2\theta d\phi^2)\right), \tag{8}$$

where $r = a\sinh\chi$, $\chi \in [0,\infty[$.

Friedmann equations

Friedmann–Lemaître–Robertson–Walker metric is a (generic) synchronous metric in a homogeneous and isotropic universe,

$$ds^2 = dt^2 - a(t)^2\left(d\chi^2 + \begin{bmatrix}\chi^2\\ \sin^2\chi\\ \sinh^2\chi\end{bmatrix}(d\theta^2+\sin^2\theta d\phi^2)\right), \tag{9}$$

where $a(t)$ is the time-dependent *scale factor* of the universe.

Closed universe

In a closed Friedmann universe the metric is

$$ds^2 = a^2\left(d\eta^2 - d\chi^2 - \sin^2\chi(d\theta^2+\sin^2\theta d\phi^2)\right), \tag{10}$$

where $r = a\sin\chi$, and η is the scaled time coordinate,

$$dt = a d\eta\,. \tag{11}$$

Now the following Maxima script

```
derivabbrev:true; /* a better notation for derivatives */
load(ctensor); /* load the package to deal with tensors */
ct_coords:[eta,chi,theta,phi]; /* our coordiantes: eta chi theta phi */
depends([a],[eta]); /* the scale, a, depends on eta */
lg:ident(4); /* set up a 4x4 identity matrix */
lg[1,1]: a^2; /* g_{\eta\eta} = a^2 */
lg[2,2]:-a^2; /* g_{\chi\chi} = -a^2 */
lg[3,3]:-a^2*sin(chi)^2; /* g_{\chi\chi} */
lg[4,4]:-a^2*sin(chi)^2*sin(theta)^2; /* g_{\phi\phi} */
cmetric(); /* compute the prerequisites for further calculations */
christof(mcs); /* print out the Christoffel symbols, mcs_{bca}=\Gamma^a_{bc} */
uricci(true); /* print out the elements of the Ricci tensor */
scurvature(); /* Ricci scalar curvature */
```

calculates the components of the Ricci tensor[2] and the Ricci scalar for this metric,

$$R^\eta_\eta = \frac{3}{a^4}(a'^2 - aa'')\,,\ R^\chi_\chi = R^\theta_\theta = R^\phi_\phi = -\frac{1}{a^4}(2a^2 + a'^2 + aa'')\,,\ R = -\frac{6}{a^3}(a + a'')\,, \tag{12}$$

where prime denotes the η-derivative.

[2]The script also calculates the Christoffel symbols, ${\Gamma^\eta}_{\eta\eta} = {\Gamma^\chi}_{\eta\chi} = {\Gamma^\theta}_{\eta\theta} = {\Gamma^\phi}_{\eta\phi} = {\Gamma^\eta}_{\chi\chi} = \frac{a'}{a}$, ${\Gamma^\theta}_{\chi\theta} = {\Gamma^\phi}_{\chi\phi} = \cot\chi$, ${\Gamma^\eta}_{\theta\theta} = \frac{a'}{a}\sin^2\chi$, ${\Gamma^\chi}_{\theta\theta} = -\cos\chi\sin\chi$, ${\Gamma^\phi}_{\theta\phi} = \cot\theta$, ${\Gamma^\eta}_{\phi\phi} = \frac{a'}{a}\sin^2\chi\sin^2\theta$, ${\Gamma^\chi}_{\phi\phi} = -\cos\chi\sin\chi\sin^2\theta$, ${\Gamma^\theta}_{\phi\phi} = -\cos\theta\sin\theta$.

The stress-energy-momentum tensor for the perfect fluid is[3]

$$T_{ab} = (\epsilon + p)u_a u_b - p g_{ab} \,, \tag{13}$$

where ϵ is the rest-energy density and p is the pressure. In synchronous Friedmann coordinates the matter is at rest and the 4-velocity is $u^b = \{\frac{1}{a}, 0, 0, 0\}$.

Correspondingly, the Einstein equation

$$R^a_b - \frac{1}{2} R \delta^a_b = \kappa T^a_b \tag{14}$$

then has the ${}^{\eta}_{\eta}$ component

$$\frac{3}{a^4}(a^2 + a'^2) = \kappa\epsilon, \tag{15}$$

and the three identical spatial components,

$$\frac{1}{a^4}(a^2 + 2aa'' - a'^2) = -\kappa p \,, \tag{16}$$

called the *Friedmann equations* for a closed universe,

If the relation between ϵ and p, called the *equation of state* of the matter, is known, the energy density ϵ can be determined as function of a from the energy conservation equation. The latter must have the form

$$dE = -p dV \,, \tag{17}$$

where V is a volume element in the Friedmann universe, and $E = \epsilon V$ is the energy content of this volume. Since the volume is proportional to a^3, and both ϵ and a in a Friedmann universe can only depend on time, equation (17) can be rewritten as

$$(\epsilon a^3)' + p(a^3)' = 0 \,. \tag{18}$$

It is easy to show, that equation (18) actually follows from the Friedmann equations (15) and (16).

The energy conservation equation (17) can also be written as

$$\frac{3da}{a} = -\frac{d\epsilon}{\epsilon + p} \,. \tag{19}$$

When the dependence $\epsilon(a)$ is found by integration of the energy conservation equation (19), the solution to the Friedmann equation can be obtained as the integral

$$\eta = \pm \int \frac{da}{a\sqrt{\frac{1}{3}\kappa\epsilon a^2 - 1}} \,. \tag{20}$$

Open universe

In an open Friedmann universe the metric is

$$ds^2 = a^2 \left(d\eta^2 - d\chi^2 - \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right) \,, \tag{21}$$

where $r = a \sinh \chi$, and $ad\eta = dt$. This metric can be obtained from the closed universe metric (10) by a formal substitution

$$\{a, \eta, \chi\} \to \{ia, i\eta, i\chi\} \,. \tag{22}$$

[3]Interpreting the equations $T^{ab}{}_{,b} = 0$ as conservation laws leads to the following interpretations of the components of the stress-energy-momentum tensor: T^{00} is energy density, $T^{0\alpha}$ is momentum density, $T^{\alpha\alpha}$ is pressure, and $T^{\alpha\beta}$ wher $\alpha \neq \beta$ is the shear stress. For a perfect fluid at rest the shear stress is zero and the momentum is also zero. Thus in the frame where the element of the liquid is at rest, $u^a = \{1, 0, 0, 0\}$, the stress-energy-momentum tensor is diagonal with components ϵ, p, p, p where ϵ is the rest-energy and p is the pressure. Apparently, the covariant form must then be $T^{ab} = (\epsilon + p)u^a u^b - p g^{ab}$.

Therefore the Friedmann equation for an open universe can be readily obtained from (15) by the substitution (22),

$$\frac{3}{a^4}(a'^2 - a^2) = \kappa\epsilon, \tag{23}$$

with the integral solution,

$$\eta = \pm \int \frac{da}{a\sqrt{\frac{1}{3}\kappa\epsilon a^2 + 1}}\,. \tag{24}$$

Flat universe

In a flat Friedmann universe the metric is

$$ds^2 = dt^2 - a(t)^2\left(dx^2 + dy^2 + dz^2\right)\,, \tag{25}$$

and the Friedmann equation is (see the exercise)

$$3\frac{\dot{a}^2}{a^2} = \kappa\epsilon\,. \tag{26}$$

Exercises

1. Our ${}^{\eta}_{\eta}$ Friedmann equation is written for the η-derivative, $a' \doteq da/d\eta$. Rewrite it (as in the Wikipedia article) for the t-derivative $\dot{a} \doteq da/dt$.
2. Calculate manually the Ricci tensor in Friedmann coordinates for a closed universe.
3. Argue that the energy conservation equation,

$$(\epsilon a^3)' + p(a^3)' = 0\,, \tag{27}$$

   follows from the field equations

$$\frac{3}{a^4}(a^2 + a'^2) = \kappa\epsilon\,, \tag{28}$$

$$\frac{1}{a^4}(a^2 + 2aa'' - a'^2) = -\kappa p\,. \tag{29}$$

4. Consider a flat (Euclidean) isotropic universe with the metric

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)\,.$$

   and investigate its temporal development for matter and radiation dominated universes. Hints:

   (a) Calculate the Christoffel symbols[4];
   (b) Calculate the Ricci tensor and the Ricci scalar[5];
   (c) Write down the ${}^{t}_{t}$ component of the Einstein equation with perfect fluid[6];

4
$$\Gamma^x{}_{tx} = \Gamma^y{}_{ty} = \Gamma^z{}_{tz} = \frac{\dot{a}}{a}\,,\ \Gamma^t{}_{xx} = \Gamma^t{}_{yy} = \Gamma^t{}_{zz} = a\dot{a}\,.$$

5
$$R^t_t = -3\frac{\ddot{a}}{a}\,,\ R^x_x = R^y_y = R^z_z = -\frac{\ddot{a}}{a} - 2\frac{\dot{a}^2}{a^2}\,.$$

6
$$3\frac{\dot{a}^2}{a^2} = \kappa\epsilon\,.$$

(d) Write down the energy conservation equation[7];

(e) Integrate the equations for a matter dominated universe ($p = 0$, $\epsilon = \mu$)[8];

(f) Integrate the equations for a radiation dominated universe ($p = \epsilon/3$)[9];

5. Calculate the volumes of the closed and open universes. Hint: in a metric space the (covariant) volume element dV is given via the determinant g of the metric tensor as

$$dV = \sqrt{|g|} dx^1 \dots dx^n \,. \tag{30}$$

[7]
$$\frac{dV}{V} = -\frac{d\epsilon}{(\epsilon + p)} \;\Rightarrow\; 3\ln(a) = -\int \frac{d\epsilon}{(\epsilon + p)} \,.$$

[8] $$\mu a^3 = \text{const}\,,\; a = \propto t^{2/3} \,.$$

[9] $$\epsilon a^4 = \text{const}, a = \propto t^{1/2} \,.$$