# Motion in Schwarzschild metric

Motion in the Schwarzschild metric reveals several of the unusual consequences of general relativity:

1. Utmost *relativity of measurements*: it takes finite proper time for a body to fall onto the Schwarzschild radius, yet for the outside observer it takes infinite time.
2. There exist *gravitational singularities* (geodesic incompleteness) in general relativity: some trajectories cannot be extended beyond a certain point. Gravitational singularities—unlike coordinate singularities—do not depend on the coordinate system and cannot be removed by coordinate transformation. The gravitational field becomes infinitely large at a gravitational singularity.
3. There exist *event horizons* in general relativity – the (hyper)surfaces in time-space which can only be crossed in one direction.

## Radial fall in Schwarzschild coordinates

Let us derive the equation of motion—in Schwarzschild coordinates $\{t,\ r\}$—for a test body falling radially from infinity towards a black hole[1] with mass $M$.

The radial equation can be conveniently obtained from the metric expression along the trajectory of the falling body,

$$ds^2 = \left(1-\frac{R}{r}\right)dt^2 - \frac{dr^2}{1-\frac{R}{r}}\ , \tag{1}$$

where $R = 2M\cdot(G/c^2)$ is the gravitational radius and where $d\theta = d\phi = 0$ for the radial motion.

We can get rid of $ds$ using the geodesic equation for the time-coordinate,

$$\frac{d}{ds}\left[\left(1-\frac{R}{r}\right)\frac{dt}{ds}\right] = 0\ , \tag{2}$$

with the first integral

$$\left(1-\frac{R}{r}\right)\frac{dt}{ds} = E\ . \tag{3}$$

For a body that starts falling from at rest at infinity $E = 1$ (prove this). Therefore along the trajectory of our body the following relation holds,

$$ds = \left(1-\frac{R}{r}\right)dt\ . \tag{4}$$

Inserting this into (1) gives the sought equation of motion,

$$\frac{dr}{dt} = -\sqrt{\frac{R}{r}}\left(1-\frac{R}{r}\right)\ . \tag{5}$$

Notice that at large distances, $r \gg R$, the equation turns into the corresponding Newtonian equation. However as the body approaches the gravitational radius its radial velocity approaches zero. Thus for the external observer the falling body never reaches the gravitational radius.

This observation (and the fact that the metric in Schwarzschild coordinates has a singularity at the gravitational radius) makes the Schwarzschild coordinates not very useful for investigations of the motion of the bodies close to gravitational radius. We shall therefore switch to an alternative set of coordinates (for the same metric) – the Lemaitre coordinates named after George Lemaitre.

[1] A black hole is a body that is completely under the gravitational radius

## Lemaitre coordinates

In the Schwarzschild metric around a body with the gravitational radius $R$,

$$ds^2 = \left(1 - \frac{R}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{R}{r}} - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) , \quad (6)$$

there is a singularity at the gravitational radius, $r = R$. Under the gravitational radius the coordinate $r$ becomes time-like and $t$ becomes space-like.

However, it turns out to be not a physical singularity, but rather an artifact of the (incorrect) assumption that a static Schwarzschild coordinates can be realized under the gravitational radius with material bodies. The singularity at the Schwarzschild radius in Schwarzschild coordinates can be removed by a coordinate transformation. Such removable singularities are called *coordinate singularities*.

George Lemaitre was apparently the first one to figure that out. He introduced an alternative set of coordinates, $\{t,\ r\} \to \{\tau,\ \rho\}$, which are connected to bodies falling free radially towards the center from infinity where they are at rest. The angular coordinates remain the same.

The transformation to the Lemaitre coordinates $\tau$, $\rho$ is given as

$$d\rho = dt + \sqrt{\frac{r}{R}} \frac{dr}{1 - \frac{R}{r}} , \quad (7)$$

$$d\tau = dt + \sqrt{\frac{R}{r}} \frac{dr}{1 - \frac{R}{r}} . \quad (8)$$

The first equation is inspired by our equation of motion (5): the body with a fixed coordinate $\rho$ satisfies (5) and therefore falls freely towards the center. The second equation is chosen such that the metric expression in the new coordinates is *synchronous* (has a term $d\tau^2$ and no cross-term $d\tau d\rho$).

The transformation to the Lemaitre coordinate gives the expression for the Schwarzschild metric where the singularity at $r = R$ is removed (however there remains a genuine singularity at the origin),

$$ds^2 = d\tau^2 - \frac{R}{r} d\rho^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) , \quad (9)$$

where $r = [\frac{3}{2}(\rho - \tau)]^{2/3} R^{1/3}$. The latter is obtained by integrating

$$d\rho - d\tau = \sqrt{\frac{r}{R}} dr , \quad (10)$$

which is the difference between (7) and (8).

The set of Lemaitre coordinates is *synchronous*[2] and is realized by a system of clocks in a free radial fall from infinity towards the origin.

## Radial fall in Lemaitre coordinates

For a free falling body $d\rho = 0$ and equation (7) reduces to equation (5),

$$dt = -\sqrt{\frac{r}{R}} \frac{1}{(1 - \frac{R}{r})} dr . \quad (11)$$

Approaching the Schwarzschild radius, in the region $r \gtrsim R$, we have in the lowest order in $(r - R)/R$,

$$dt = -\frac{R}{r - R} dr , \quad \Rightarrow \quad \frac{r - R}{r_1 - R} = e^{-\frac{t - t_1}{R}} . \quad (12)$$

[2] the metric has the form $ds^2 = d\tau^2 + g_{\alpha\beta} dx^\alpha dx^\beta$.

Apparently, it takes a free falling body infinitely long $t$-time—the time used by the outer observer—to reach the Schwarzschild radius.

On the contrary, a free falling Lemaitre clock moves from some radius $r_1$ to a smaller radius $r_2$—which can well be the gravitational radius or even the origin—within finite $\tau$-time $\Delta\tau_{12}$. Indeed, setting $d\rho = 0$ in (10) gives

$$\Delta\tau_{12} = -\int_{r_1}^{r_2} \sqrt{\frac{r}{R}} dr = \frac{2}{3} \left( \frac{r_1^{3/2} - r_2^{3/2}}{R^{1/2}} \right) . \tag{13}$$

## Event horizons and black holes

Along the trajectory of a radial light ray the infinitesimal interval is zero,

$$ds^2 = d\tau^2 - \frac{R}{r} d\rho^2 = 0 \,, \tag{14}$$

which gives

$$d\rho = \pm\sqrt{\frac{r}{R}} d\tau \ , \tag{15}$$

where plus and minus describe the rays of light sent correspondingly up and down.

Isolating $d\rho$ in (10) and inserting the result into (15) shows that along the trajectory

$$dr = \left( \pm 1 - \sqrt{\frac{R}{r}} \right) d\tau \,. \tag{16}$$

Apparently if $r < R$ then there is always $dr < 0$ and thus the light rays emitted radially inwards and outwards both end up at the origin. In other words no signal can escape from inside the gravitational radius. This phenomenon is called the event horizon.

Therefore a massive object with a size less than the gravitational radius, called a black hole, is completely under the event horizon and its interior is totally invisible.

The trajectories of massive bodies and light rays inside the gravitational radius both end up in the origin where they cannot be extended any further.

The black holes can possibly be detected through their interaction with the matter outside the event horizon.

## Exercises

1. In Newtonian mechanics derive the equation of motion for a body falling radially from infinity towards the gravitating center of mass $M$. Compare with the relativistic result.

2. Show that in a *synchronous* reference frame, where the metric has the form

   $$ds^2 = d\tau^2 + g_{\alpha\beta} dx^\alpha dx^\beta \big|_{\alpha,\beta=1,2,3}$$

   the time lines (where $dx^\alpha = 0$) are geodesics.

3. In Schwartzschild metric derive the equation of motion (11),

   $$dt = -\sqrt{\frac{r}{R}} \frac{1}{(1 - \frac{R}{r})} dr \ ,$$

   for a body in a radial free fall from infinity towards a black hole using the expression for the metric and the geodesic equation for the time-coordinate.

4. In Schwartzschild metric Derive the equation of motion (11),

$$dt = -\sqrt{\frac{r}{R}}\frac{1}{(1-\frac{R}{r})}dr \ ,$$

for a body in a radial free fall from infinity towards a black hole using the geodesic equation for the radial coordinate.

Hints: find $dt/ds$ from the $t$ geodesic equation; find $dr/ds$ from the $r$ geodesic equation eliminating $t(s)$ using the expression for the metric and integrating once; divide the two. [3]

5. In the Schwarzschild metric a body is falling free radially toward the center. What is its coordinate velocity $dr/dt$ at radius $r$? What is its locally measured velocity at the same place? Hints: in the Schwarzschild metric the locally measured radial length is given as $d\hat{r}^2 = (1-\frac{R}{r})^{-1}dr^2$ and the locally measured time is given as $d\hat{t}^2 = (1-\frac{R}{r})dt^2$.

6. A radio transmitter is free falling radially toward a black hole. When the transmitter is approaching the gravitational radius an outside observer measures its radio signal to be red-shifted as $\omega = \omega_0 e^{-\lambda t}$. Estimate the mass of the black hole from the measured $\lambda$.

   Hints: $\omega = \omega_0\sqrt{g_{00}}$; $r - R = (r_1 - R)e^{-(t-t_1)/R}$ (see equation (12)).

7. Calculate the proper time it takes for a Lemaitre clock to fall from the gravitational radius to the center of a black hole. For a black hole with the solar mass specify this time in seconds.

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[3] The radial geodesic equation, $\frac{du_r}{ds} = \frac{1}{2}g_{bc,r}u^b u^c$, for the radial motion ($u^\theta = u^\phi = 0$) in the Schwartzschild metric is given as

$$\frac{d}{ds}\left(-\frac{1}{1-\frac{R}{r}}\dot{r}\right) = \frac{1}{2}\left(1-\frac{R}{r}\right)_{,r}\dot{t}^2 + \frac{1}{2}\left(-\frac{1}{1-\frac{R}{r}}\right)_{,r}\dot{r}^2 \ , \tag{17}$$

where $\dot{}$ denotes $\frac{d}{ds}$ and $R \equiv R$ is the gravitational radius of the central body. We can eliminate $\dot{t}^2$ from the equation using the expression for the metric for the radial motion,

$$ds^2 = \left(1-\frac{R}{r}\right)dt^2 - \frac{1}{1-\frac{R}{r}}dr^2 \ , \tag{18}$$

or, equivalently,

$$\dot{t}^2 = \frac{1+\frac{1}{1-\frac{R}{r}}\dot{r}^2}{1-\frac{R}{r}} \ . \tag{19}$$

Substituting this into the radial equation gives (after few identical transformations),

$$\ddot{r} = -\frac{1}{2}\frac{R}{r^2} \ , \tag{20}$$

which formally is the Newtonian equation. It can be integrated once by multiplying with $\dot{r}$, which gives

$$\frac{d}{ds}\dot{r}^2 = \frac{d}{ds}\left(\frac{R}{r}\right) \ , \tag{21}$$

and (explain)

$$\frac{dr}{ds} = -\sqrt{\frac{R}{r}} \ . \tag{22}$$

Now, the $t$-equation reads

$$\frac{d}{ds}\left(\left(1-\frac{R}{r}\right)\dot{t}\right) = 0 \ , \tag{23}$$

with the first integral

$$\left(1-\frac{R}{r}\right)\dot{t} = E = 1 \ , \tag{24}$$

which gives

$$\frac{dt}{ds} = \frac{1}{1-\frac{R}{r}} \ . \tag{25}$$

Finally,

$$\frac{dr}{dt} = \frac{(\frac{dr}{ds})}{(\frac{dt}{ds})} = -\sqrt{\frac{R}{r}}\left(1-\frac{R}{r}\right) \ . \tag{26}$$

8. Show that in Newtonian mechanics the circular planetary orbits around stars are stable against small radial perturbations. Show that in general relativity circular orbits are stable only if $r > 6M$. Hints: consider a circular orbit with a small perturbation, $u = u_0 + \delta u$; derive the equation for $\delta u$ in the lowest order; investigate whether the perturbation remains small or increases.

9. Show that equatorial orbits in the Schwarzschild metric are stable. Consider for simplicity a circular orbit. Hint: consider an equatorial orbit with a small perturbation, $\theta = \pi/2 + \delta\theta$; derive the equation for $\delta\theta$ in the lowest order; show that the perturbation remains small; interpret the solution.