# Prologue

*General relativity* is a classical relativistic theory of gravitation. It was introduced by Albert Einstein in 1916. General relativity is the accepted description of gravitation in modern physics.

General relativity is a *geometric* theory: the gravitational field—unlike the electromagnetic field—is not a material field but rather a curvature of space-time. In general relativity massive bodies do not create a material field around them—like the chares do in electrodynamics—but rather distort the space-time in their vicinity which affects the motion of other bodies.

General relativity satisfies the correspondence principle[1]: in the absence of gravitational fields general relativity reduces to special relativity, and in the limit of weak gravitational fields and non-relativistic velocities general relativity reduces to Newtonian gravitation.

Although not the only relativistic theory of gravitation, general relativity is the simplest theory consistent with experimental data.

General relativity has important astrophysical implications and is the basis of the current cosmological models of the universe.

Unlike classical electrodynamics general relativity has not been quantized – a complete and self-consistent theory of quantum gravity does not exist yet.

# A brief reminder on Special Relativity

*Special relativity*—formulated by Albert Einstein in 1905—is a theory of spatial and temporal measurements in inertial frames of reference. And it is also the theory of relativistic kinematics[2]. Special relativity is the basis of relativistic mechanics and electrodynamics. It establishes, in particular, the law of coordinate transformations between inertial frames—the Lorentz transformation—as well as the formulae for the relativistic momentum and the relativistic energy of moving bodies. In the slow motion limit special relativity reduces to Galilean relativity.

Special relativity is based on the following postulates:[3]

1. **Existence of inertial frames of reference**: in the absence of gravitational forces there exist infinitely many *inertial frames of reference* where the laws of physics take their simplest form (as indeed no inertial forces are present in inertial frames). In particular, free bodies—that is, bodies not affected by forces—move with constant velocities along straight lines. In an inertial frame one can conveniently introduce Cartesian coordinates[4] where the geometry of space-time is particularly simple (Minkowski space-time).

   Inertial frames move with constant velocities with respect to each other and measurements in one inertial frame can be converted to measurements in another by a linear transformation (the Lorentz transformation).

2. **Special principle of relativity**: the laws of physics have the same form in all inertial frames.

3. **Homogeneity and isotropy of space**: the space has the same properties at any place and in any direction.

[1]The correspondence principle suggests that a new theory should reproduce the results of the older well-established theories in those domains where the old theories are applicable.

[2]*Kinematics* deals with the motion of free bodies (bodies that not affected by forces). The study of how forces affect the motion of bodies falls within *dynamics* (or *kinetics*, in old textbooks).

[3]In physics, a *postulate* is a physical law of a more general nature which is typically deduced from a large number of different experiments.

[4]where the distance $dl$ between two infinitesimally close points separated by $\{dx, dy, dz\}$ is given as

$$dl^2 = dx^2 + dy^2 + dz^2$$

4. **Finite maximum speed of a physical object**: the maximum speed with which a physical object can travel relative to a physical observer is finite (and relatively small, 299792458 m/s).

   Equivalently, one can rather postulate—as Einstein originally did—the **constancy of the speed of light**, as motivated by Maxwell's theory of electromagnetism and the null result of the Michelson–Morley experiment.

## Lorentz transformation

Lorentz transformation relates the measurements of spatial and temporal intervals in different inertial frames. It is a linear transformation between (Euclidean) coordinates in two frames as it transforms a linear motion of a free body in one inertial frame to a linear motion of the the same body in another frame.

Coordinate transformations between inertial fames form a *group*[5].

Let us consider a linear transformation of coordinates between two inertial frames (with parallel Cartesian coordinates) moving with relative velocity $v$ along the $x$-axis[6]. There exist only one form of this transformation that is consistent with isotropy of space and the group postulates. It is given as

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \begin{bmatrix} 1 & -\frac{v}{c^2} \\ -v & 1 \end{bmatrix} \begin{pmatrix} t \\ x \end{pmatrix}, \tag{1}$$

where $(t', x')$ are the coordinates in the frame $K'$ which moves relative to the frame $K$ with coordinates $(t, x)$ with velocity $v$ along the $x$ (and $x'$) axis. The $y$- and $z$-coordinates, perpendicular to the velocity boost, transform identically and are therefore omitted for brevity.

The velocity $c$ is a universal constant, the fastest possible relative velocity of two inertial frames. It equals the speed of light in vacuum and is experimentally measured to be finite.

Transformation (1) with finite $c$ is called the Lorentz transformation. Note that time and space in Lorentz transformations do not transform separately but rather as components of one inseparable four-component space-time point $x^a \doteq (t, x, y, z)$.

In the non-relativistic limit, $c \to \infty$, the Lorentz transformation turns into Galilean transformation,

$$\begin{aligned} t' &= t \,, \\ x' &= x - vt \,. \end{aligned} \tag{2}$$

Here time is absolute and does not transform at all. The time-space coordinates then separate into invariant time and the Eucledean vector of three spatial coordinates.

## Invariant space-time interval and metric

A direct calculation shows that the infinitesimal *space-time interval*,

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \,, \tag{3}$$

[5]In mathematics, a *group* is a set of elements, $G = \{a, b, c, \dots\}$, together with an operation, $*$, that combines any two of its elements to form a third element also in the set while satisfying four conditions called the group axioms, namely *closure*,

$$\forall a, b \in G : a * b \in G \,,$$

*associativity*,

$$(a * b) * c = a * (b * c) \,,$$

*identity*

$$\exists \mathrm{I} \in G : \forall a : a * \mathrm{I} = a \,,$$

and *invertibility*

$$\forall a \in G \; \exists a^{-1} \in G : a * a^{-1} = \mathrm{I} \,.$$

[6]This transformation is often called *Lorentz boost*, or *velocity boost*, or simply *boost*.

is invariant under the Lorentz transformation (1). It thus defines a *metric*[7]. A space with a metric is called *metric space*.

The pseudo[8]-Euclidean metric (3) is called *Minkowski metric* and a space with such metric is called *Minkowski space*.

The existence of a metric allows development of a geometry of space: measurements of distances, angles, and time intervals. However, geometry in Minkowski space is sometimes different from the everyday Euclidean geometry. In particular, distances and time intervals—unlike the invariant space-time interval—are relative: they might take different values in different frames.

In the limit $v \ll c$ Minkowski space reduces to *Euclidean space*, which is the non-relativistic world of classical mechanics with Galilean transformation where $dt$ is itself invariant and the Minkowski metric reduces to the Euclidean metric,

$$dl^2 = dx^2 + dy^2 + dz^2 . \tag{4}$$

## Four-vector notation

In special relativity the set of four space-time coordinates $\{t, \vec{r}\}$ transform linearly from one inertial frame to another (that is, under rotations and velocity-boosts). The coordinate differentials (infinitesimal differences) $\{dt, d\vec{r}\}$ of course also transform linearly. It is then convenient to introduce the four-vector notation,

$$dx^a \doteq \{dt, d\vec{r}\},\ a = 0, 1, 2, 3 , \tag{5}$$

where the Lorentz transformation is written in the form

$$dx'^a = \sum_{b=0}^{3} \Lambda^a_b dx^b , \tag{6}$$

where $\Lambda^a_b$ is the Lorentz transformation matrix (a product of rotation and velocity-boost matrices). In special relativity not only coordinate differentials but also the coordinates themselves transform linearly so we could just as well use $x^a$ instead of $dx^a$. However, in general relativity the coordinates generally do not transform linearly, only their differentials.

Now, any set of four numbers, $A^a$ where $a = 0, 1, 2, 3$, which under Lorentz transformation transform the same way as coordinate differential,

$$A'^a = \sum_{b=0}^{3} \Lambda^a_b A^b , \tag{7}$$

is called a four-vector.

The (invariant) metric $ds^2 = dt^2 - d\vec{r}^2$ cannot be conveniently written with a summation ($\sum$) formula (since there are three minuses and one plus there), therefore one needs to introduce a metric tensor (this particular one is called Minkowski metric tensor),

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{8}$$

[7] A *metric* is a function that defines a distance between two infinitesimally close points in a space. Metric can be used to measure distances and angles which allows development a geometry of the space.

[8] *Euclidean metric* in an $n$-dimensional space has the form

$$ds^2 = dx_1^2 + \cdots + dx_n^2,$$

while *pseudo-Euclidean metric* has one or more negative signs,

$$ds^2 = dx_1^2 + \cdots + dx_k^2 - dx_{k+1}^2 - \cdots - dx_n^2.$$

The metric can then be written as

$$ds^2 = dt^2 - d\vec{r}^2 = \sum_{a=0}^{3}\sum_{b=0}^{3} dx^a g_{ab} dx^b \equiv g_{ab} dx^a dx^b \ . \tag{9}$$

where the last term demonstrates the so called Einstein's implicit summation notation: if there is an index which appears in both the sub- and super-position there is implicit summation over this index from 0 to 3.

Since the metric tensor is not a unity matrix we have two types of vectors, the vector with index up, $dx^a$, and the vector with index down,

$$dx_a = g_{ab} dx^b = \{dt, -d\vec{r}\} \ . \tag{10}$$

The metric can then be written

$$ds^2 = g_{ab} dx^a dx^b = dx^a dx_a \ . \tag{11}$$

The inverse metric tensor, $g^{ab} = (g_{ab})^{-1}$, defined via

$$g^{ab} g_{bc} = \delta^a_c \doteq \begin{cases} 1, \ a = c \\ 0, \ a \neq c \end{cases} \tag{12}$$

transforms an index-down vector back into index-up vector,

$$x^a = g^{ab} x_b \ . \tag{13}$$

## Relativistic momentum and energy of a massive body

The postulate that free bodies move along straight lines can be conveniently formulated through the variational (also called *least action* or *stationary action*) principle[9]. Indeed a straight line between two points is the curve with extremal measure. The measure $\mu$ of a curve in a metric space is given by the integral

$$\mu = \int ds \tag{14}$$

taken along the curve. The free bodies thus move along curves with extremal measure or, equivalently, along curves with vanishing variation of the measure,

$$\delta \int ds = 0 \ . \tag{15}$$

The postulate about the motion of free bodies can then be reformulated as a least action principle with the action

$$\mathcal{S} = \alpha \int ds \ , \tag{16}$$

where the constant $\alpha$ can be deduced from the correspondence principle: in the non-relativistic limit the action of a free body has to take the classical form, namely the temporal integral over the kinetic energy of the body,

$$\mathcal{S} \xrightarrow{v \ll c} \int dt \frac{mv^2}{2} + \text{Const} \ , \tag{17}$$

where $m$ is the mass of the body.

[9] *Action* is a (real scalar) function of the trajectory of a physical system. The trajectory actually taken by a physical system gives the minimum value of the system's action.

Calculating the non-relativistic limit of (16),

$$\mathcal{S} = \alpha \int c dt \sqrt{1 - \frac{v^2}{c^2}} \overset{v \ll c}{\longrightarrow} \alpha c \int dt \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) , \tag{18}$$

and comparing with (17) gives $\alpha = -mc$,

$$\mathcal{S} = -mc \int ds = -mc^2 \int dt \sqrt{1 - \frac{\vec{v}^2}{c^2}} . \tag{19}$$

The Lagrangian[10] $\mathcal{L}$ of a free body is thus given as

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{\vec{v}^2}{c^2}} . \tag{20}$$

From the Lagrangian one can obtain in the usual way the momentum $\vec{p}$,

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} , \tag{21}$$

and the energy $\mathcal{E}$,

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial \vec{v}}\vec{v} - \mathcal{L} = \frac{mc^2}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} , \tag{22}$$

of the body.

## Exercises

1. Argue that the transformations that connect the (Euclidean) coordinates of events in different inertial frames
   (a) are linear;
   (b) form a group where the group operation is simply performing one transformation after another[11].

2. Argue that matrices of the Galilean velocity boost (in $x$-direction, for simplicity),

$$G(v) = \begin{bmatrix} 1 & 0 \\ -v & 1 \end{bmatrix} , \tag{23}$$

[10] If the action of a system can be written as a temporal integral,

$$S = \int \mathcal{L} dt ,$$

the object $\mathcal{L}$ is called the *Lagrangian* of the system.

[11] In mathematics, a *group* is a set of elements, $G = \{a, b, c, \dots\}$, together with an operation, $*$, that combines any two of its elements to form a third element also in the set while satisfying four conditions called the group axioms, namely *closure*,

$$\forall a, b \in G : a * b \in G ,$$

*associativity*,

$$(a * b) * c = a * (b * c) ,$$

*identity*

$$\exists \mathrm{I} \in G : \forall a : a * \mathrm{I} = a ,$$

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form a mathematical group with group operation being matrix multiplication. Specifically,

$$G(v_1)G(v_2) = G(v_1+v_2)\ , \quad (24)$$

$$\Big(G(v_1)G(v_2)\Big)G(v_3) = G(v_1)\Big(G(v_2)G(v_3)\Big)\ , \quad (25)$$

$$1 = G(0)\ , \quad (26)$$

$$G(v)^{-1} = G(-v)\ . \quad (27)$$

Argue that it is a Lie group.

3. Argue that the matrices of the Lorentz transformation (say, in $x$-direction) form a group and derive the relativistic law of addition of velocities.

4. Derive the Lorentz transformation
   (a) from isotropy of space, group postulates, and finite maximum velocity;
   (b) from isotropy of space and invariance of the speed of light;
   (c) any other way.

5. Show that in Minkowski space the finite interval, $\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$, is also invariant.

6. Use the Lorentz transformation to derive i) the time dilation and ii) the length contraction formulae. Do this by identifying the pairs of events where the time or space separations are to be compared and then apply the Lorentz transformation.

7. A particle that follows the line $x = ct$ in a given frame $K$ moves with the speed of light along the $x$-axis.
   (a) Based upon that fact, what do you anticipate for the equation of that line in a frame $K'$ that moves along the $x$-axis relative the $K$-frame?
   (b) Verify your prediction using the Lorentz transformation.

8. Show that the action of a body in the form

$$\mathcal{S} = \int \mathcal{L}(\vec{r}, \vec{v})dt \quad (28)$$

leads—through the variational principle that demands $\delta\mathcal{S} = 0$ on the actual trajectory of the body—to the (Euler-Lagrange) equation of motion,

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \vec{v}} = \frac{\partial \mathcal{L}}{\partial \vec{r}}\ . \quad (29)$$

9. Consider a non-relativistic body with mass $m$ moving in a potential $V(\mathbf{r})$. Show that the Lagrangian

$$\mathcal{L} = \frac{mv^2}{2} - V(\mathbf{r})$$

leads to the normal Newton's equations of motions.

10. Argue that in special relativity a body with action $\mathcal{S} = -mc\int ds$ moves along a straight line.

11. Momentum $\vec{p}$ is the quantity that conserves along the trajectory of the body if the Lagrangian does not depend explicitly on $\vec{r}$ (through the Noether's Theorem). Argue that

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \vec{v}}\ . \quad (30)$$

12. Energy $\mathcal{E}$ is the quantity that conserves along the trajectory of the body if the Lagrangian does not depend explicitly on time (through the Noether's Theorem).

    In this case the variation of the Lagrangian under the infinitesimal transformation $t \to t + \delta t$ is given as

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\vec{r}}\delta\vec{r} + \frac{\partial\mathcal{L}}{\partial\vec{v}}\delta\vec{v} \ . \tag{31}$$

    Show that on the trajectory this can be written as the energy conservation law,

$$\delta\mathcal{E} = 0 \ , \tag{32}$$

    with the energy

$$\mathcal{E} = \frac{\partial\mathcal{L}}{\partial\vec{v}}\vec{v} - \mathcal{L} \ . \tag{33}$$

13. Consider the motion of a particle with charge $e$ and mass $m$ in a constant uniform electric field $\vec{E}$ which is, say, in the direction of the $x$-axis.

    (a) Suppose that at $t = 0$ the particle was at rest, $\vec{v} = 0$, with the coordinate $\vec{r} = 0$. Find $x(t)$.

    (b) Suppose that at $t = 0$ the particle had $\vec{r} = 0$ and $v_x = 0$, but $v_y \neq 0$. Find $x(t)$, $y(t)$ and $x(y)$.

    (c) Consider the limits $eEt \ll mc$ and $eEt \gg mc$.

    Hints:

    (a) The equation of motion of a charged particle in an electro-magnetic field $\vec{E}$, $\vec{H}$ is

$$\frac{d\vec{p}}{dt} = e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{H}\right) \ , \tag{34}$$

    where the (relativistic) momentum $\vec{p}$ and the velocity $\vec{v}$ are related as

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}} \ . \tag{35}$$

    (b)

$$\int_0^t \frac{a\tau d\tau}{\sqrt{1 + (a\tau/c)^2}} = \frac{1}{2}\frac{c^2}{a}\int_0^{(at/c)^2} \frac{d\xi}{\sqrt{1+\xi}} \tag{36}$$

$$= \frac{c^2}{a}\sqrt{1+\xi}\Big|_0^{(at/c)^2} \tag{37}$$

$$= \frac{c^2}{a}\left(\sqrt{1 + (at/c)^2} - 1\right) \ . \tag{38}$$

14. A traveler starts from Earth and moves along a line with constant acceleration $g$ for 25 traveler's years then with constant deceleration $g$ again for 25 traveler's years. How far from Earth will they reach? What was their maximum speed in the Earth's frame (assumed inertial)? The traveler then flies back to Earth in the same manner. How many years will have passed on Earth since their departure when they comes back to Earth?