Equivalence principle

General relativity is based on Einstein's *equivalence principle*. The principle postulates local equivalence of inertial and gravitational forces. Because of that principle general relativity appears as a geometric theory: massive bodies distort the geometry of space-time in their vicinity which influences the motion of other bodies. This is different to the electromagnetic forces where charged bodies do not alter the space-time geometry but rather create the electromagnetic field around them which then affects the motion of other charged bodies.

Inertial forces

Inertial forces are apparent forces that act on all bodies in non-inertial frames of reference such as a rotating frame (centrifugal and Coriolis forces) or a uniformly accelerating frame (elevator force). Inertial forces do not arise from any physical interaction but rather from the non-linear motion of the frame itself.

In an inertial frame a free body moves without acceleration, and its (non-relativistic) equation of motion in Cartesian coordinates is

$$\ddot{\vec{r}} = 0, \tag{1}$$

where $\vec{r} \doteq \{x, y, z\}$ are the three spatial coordinates of the body and the dot above a letter denotes time derivative. This is an equation for a linear motion: in inertial frames free bodies move without acceleration along straight lines.

In a non-inertial uniformly accelerating frame that moves with acceleration $\vec{a}$ relative to an inertial frame the corresponding equation of motion for the free body is

$$\ddot{\vec{r}} = -\vec{a}. \tag{2}$$

This is an equation for a curve (a parabola, I believe): in non-inertial frames free bodies generally move with acceleration along curves.

Equation (2) can be written in the form of the second Newton's law,

$$m\ddot{\vec{r}} = \vec{F}_I, \tag{3}$$

where m is the mass of the body, and

$$\vec{F}_I = -m\vec{a} \tag{4}$$

is the inertial force (in this case often called *elevator force*).

Inertial forces have the following properties:

1. Under inertial forces all massive bodies move with the same acceleration independent of their properties.
2. Inertial forces arise from the (geometrical) properties of the non-inertial frame rather than due to physical fields affecting the bodies.
3. Inertial forces completely disappear after a coordinate transformation to an inertial frame.

Equivalence of inertial and gravitational masses

Newton's law of gravitation states that the gravitational force $\vec{F}_G$ exerted on a particle by gravitating bodies is proportional to the mass m of the particle,

$$\vec{F}_G = m\vec{g}\ , \tag{5}$$

where the gravitational acceleration $\vec{g}$ depends on the positions and masses of the gravitating bodies. Therefore under gravitational forces all particles move with the same acceleration (just like under inertial forces) as the mass disappears from the second Newton's law of motion,

$$m\ddot{\vec{r}} = m\vec{g}\ . \tag{6}$$

The masses in the left- and right-hand sides of this equation are often referred to separately as *inertial* and *gravitational masses.*

Many experiments attempted to observe the difference between gravitational and inertial masses, including Galileo's measurements of acceleration of balls of different composition rolling down inclined planes, and Newton's measurements of the period of pendulums with different mass but identical length. All experiments reported no observed difference, with the most precise experiment to day (arXiv:0712.0607) having the accuracy of one part in 10^{13}.

There are theories of gravitation where the equality of gravitational and inertial masses is just a coinsidence but in general relativity this is a fundamental postulate.

Einstein's equivalence principle

Einstein has postulated that gravitational forces are locally[1] equivalent to inertial forces, that is, gravitational forces are unlike other physical forces but much like geometrical inertial forces. This postulate is called the Einstein's equivalence principle. It can be formulated in several ways:

1. Gravitational forces are locally equivalent to inertial forces.
2. An accelerated frame is locally equivalent to a frame in a uniform gravitational field.
3. Gravitational field is locally equivalent to a non-inertial frame.
4. In free fall all effects of gravity disappear in all possible local experiments and general relativity reduces locally to special relativity.

The principle is customarily illustrated by two spacecrafts, one on Earth, the other accelerating with the Earth's gravitational acceleration g in the outer space. The observers in these spacecrafts can not determine by doing local experiments inside spacecrafts whether their craft is accelerating or at rest in a gravitational field.

However, unlike inertial forces gravitational forces vanish at large distances from the sources of gravitation and therefore non-local experiments can well distinguish between inertial and gravitational forces.

The theory becomes geometric

The equivalence principle leads to the following phenomenon: the gravitational filed becomes the metric of the space and makes the space curved.

We shall observe this in the Newtonian limit. Consider a body with mass m moving in a Newtonian gravitational potential $\phi(\mathbf{r})$. The non-relativistic action is given as

$$\mathcal{S} = \int \left(\frac{mv^2}{2} - mc^2 - m\phi \right) dt \; . \tag{7}$$

This can be viewed—just as we did for a free body—as the non-relativistic limit of

$$\mathcal{S} = -mc \int \sqrt{\left(1 + \frac{2\phi}{c^2}\right) c^2 dt^2 - d\mathbf{r}^2} = -mc \int ds \; . \tag{8}$$

Therefore one can conclude that in a (weak) gravitational potential ϕ the metric of the space becomes (approximately)

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right) c^2 dt^2 - d\mathbf{r}^2 \; . \tag{9}$$

This is clearly a metric of a non-Euclidean space as, apparently, time runs differently at different depths in a gravitational potential.

[1] *Locally* means within a limited space, where variations of the fields can be neglected.

Flat and curved spaces

A space with Euclidean or pseudo-Euclidean metric is called *flat*. For example, the Minkowski space of special relativity is flat.

The coordinates in which the metric is (pseudo)-Euclidean are also often called flat. If the metric is not everywhere (pseudo)-Euclidean, the coordinates are called *curvilinear*. For example, in a flat two-dimensional space with polar coordinates $\{r, \theta\}$ the metric is non-Euclidean,

$$dl^2 = dr^2 + r^2 d\theta^2. \tag{10}$$

However, if there exist a coordinate transformation that globally turns the metric into (pseudo)-Euclidean, the space is still called flat. In our example such transformation is

$$\begin{cases} x = r\cos\theta \ , \\ y = r\sin\theta \ . \end{cases} \tag{11}$$

If such transformation does not exist, the space is called *curved* or *Riemann space*. The geometry of a curved space is called non-Euclidean geometry or *Riemann geometry*.

Space-time in a gravitational field is curved

Einstein's equivalence principle states that gravitational forces disappear locally in the frame of a free falling observer. A free falling observer finds themselves in a local Minkowski space-time.

In other words the space-time in a gravitational field can be transformed locally to Minkowski space-time by a non-linear coordinate transformation.

Yet in other words the space-time in a gravitational field locally can be obtained by a non-linear coordinate transformation from a Minkowski space-time.

This transformation cannot be global though since gravitational forces, unlike inertial forces, vanish at large distances from massive bodies.

Therefore the consequence of Einstein's equivalence principle is that the space-time in a gravitational field is genuinely curved.

General principle of relativity

In the presence of gravitational fields the space-time is curved, and it is not possible to build a set of globally flat coordinates. Therefore any frame of reference with arbitrary curvilinear coordinates and arbitrarily tuned clocks must be equally accepted in general relativity[2].

The general principle of relativity can then formulated as

The laws of physics should have the same form in arbitrary frames of reference.

In order to build suitable differential equations, invariant under general coordinate transformations, we need to develop differential geometry in curvilinear coordinates.

Exercises

1. A pendulum is suspended from the roof of a car moving in a line (straight curve) with constant acceleration a. Find the angle the pendulum makes with the vertical. Explain what is happening from the viewpoint of an inertial observer outside of the car and a non-inertial observer in the car.

2. A bucket of water slides freely under gravity down a slope of a fixed angle α to the horizontal. What is the angle of inclination of the surface of water relative to the base of the bucket?

3. Newtonian gravitation

[2] The accepted coordinates must be differentiable (to allow development of differential geometry) and the coordinate transformations must be invertible.

(a) Argue that the equation of motion of a test body with coordinate $\vec{r}$ in Newton's theory of gravitation,

$$\ddot{\vec{r}} = -\sum_k \frac{G_N M_k}{|\vec{r}-\vec{r}_k|^2} \frac{\vec{r}-\vec{r}_k}{|\vec{r}-\vec{r}_k|} , \tag{12}$$

where M_k and $\vec{r}_k$ are the masses and coordinates of gravitating bodies, can be written as

$$\ddot{\vec{r}} = -\vec{\nabla}\phi , \tag{13}$$

where ϕ is the gravitational potential,

$$\phi(\vec{r}) = -\sum_k \frac{G_N M_k}{|\vec{r}-\vec{r}_k|} . \tag{14}$$

(b) Argue that the gravitational potential satisfies the equation[3].

$$\nabla^2\phi(\vec{r}) = 4\pi G_N \mu(\vec{r}) , \tag{15}$$

where $\mu(\vec{r})$ is the mass density of gravitating bodies,

$$\mu(\vec{r}) = \sum_k M_k \delta(\vec{r}-\vec{r}_k) . \tag{16}$$

(c) Argue that the equation of motion (12) and the equation for the gravitational potential (15) are invariant under Galilean transformation.

4. Gravitation as geometry of space

   (a) Argue that the non-relativistic motion of a test body with mass m in a Newtonian gravitational potential ϕ can be described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}mv^2 - m\phi - mc^2 , \tag{17}$$

   where v is the velocity of the body.

   (b) Argue that this Lagrangian can be obtained as the weak-field non-relativistic limit ($v \ll c$ and $\phi \ll c^2$) from the action $S = -mc\int ds$ where the infinitesimal interval ds is given as

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right) c^2 dt^2 - d\vec{r}^2 . \tag{18}$$

   (c) Argue that this metric describes a curved (non-Euclidean) space and that the proper time runs differently at different levels in the gravitational potential.

5. Explain the "twins paradox" of special relativity from the viewpoints of i) the inertial twin and ii) the non-inertial (travelling) twin.

6. What is the path of a free particle in a uniform gravitational field?

7. Argue that equivalence principle implies bending of light in gravitational fields.

8. Argue that equivalence principle implies gravitational redshift.

9. Construct an experiment to find out whether one is in an inertial frame.

10. Are the following coordinate transformations $(x, y) \to (x', y')$ allowed in general relativity?

    (a) $x' = x,\ y' = 1$
    (b) $x' = \sqrt{x^2 + y^2},\ y' = \arctan(y/x)$
    (c) $x' = 1/\sqrt{x^2 + y^2},\ y' = \arctan(y/x)$
    (d) $x' = \ln(x),\ y' = y$

[3] Hint: $\nabla^2 \frac{1}{r} = -4\pi\delta(\vec{r})$.