Motion in Schwarzschild metric

Motion in the Schwarzschild metric reveals several of the unusual consequences of general relativity:

1. Utmost *relativity of measurements*: it takes finite proper time for a body to fall onto the Schwarzschild radius, yet for the outside observer it takes infinite time;

2. There exist *gravitational singularities* (geodesic incompleteness) in general relativity: some trajectories cannot be extended beyond a certain point. Gravitational singularities—unlike coordinate singularities—do not depend on the coordinate system and cannot be removed by coordinate transformation. The gravitaional field becomes infinitely large at a gravitational singularity.

3. There exist *event horizons* in general relativity – the hyper-surfaces in time-space which can only be crossed in one direction.

Lemaitre coordinates

In the Schwarzschild metric around a body with the gravitational radius r_g,

$$ds^2 = (1 - \frac{r_g}{r})dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) , \tag{1}$$

there is a singularity at the gravitational radius, $r = r_g$. Under the gravitational radius the coordinate r becomes time-like and t becomes space-like.

However, it turns out to be not a physical singularity, but rather an artifact of the (incorrect) assumption that a static Schwarzschild coordinates can be realized under the gravitational radius with material bodies. The singularity at the Schwarzschild radius in Schwarzschild coordinates can be removed by a coordinate transformation. Such removable singularities are called *coordinate singularities*.

A transformation to the Lemaitre coordinates τ, ρ

$$d\tau = dt + \sqrt{\frac{r_g}{r}} \frac{dr}{1 - \frac{r_g}{r}} , \tag{2}$$

$$d\rho = dt + \sqrt{\frac{r}{r_g}} \frac{dr}{1 - \frac{r_g}{r}} , \tag{3}$$

leads to the Lemaitre coordinate expression for the Schwarzschild metric, where the singularity at r_g is removed[1],

$$ds^2 = d\tau^2 - \frac{r_g}{r} d\rho^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) , \tag{4}$$

where $r = [\frac{3}{2}(\rho - \tau)]^{2/3} r_g^{1/3}$. The latter is obtained by integrating

$$d\rho - d\tau = \sqrt{\frac{r}{r_g}} dr , \tag{5}$$

which is the difference between (3) and (2).

The Lemaitre coordinates are synchronous[2] and are thus realized by a system of clocks in a free radial fall from infinity towards the origin.

[1] there remains a genuine singularity at the origin.
[2] the metric has the form $ds^2 = d\tau^2 + g_{\alpha\beta} dx^\alpha dx^\beta$.

Radial fall towards the origin

For a free falling body $d\rho = 0$ and equation (3) gives

$$dt = -\sqrt{\frac{r}{r_g}}\frac{1}{(1-\frac{r_g}{r})}dr \, . \tag{6}$$

Approaching the Schwarzschild radius, in the region $r \gtrsim r_g$, we have in the lowest order in $(r - r_g)/r_g$,

$$dt = -\frac{r_g}{r - r_g}dr \, , \quad \Rightarrow \quad \frac{r - r_g}{r_1 - r_g} = e^{-\frac{t-t_1}{r_g}} \, . \tag{7}$$

Apparently, it takes a free falling body infinitely long t-time—the time used by the outer observer—to reach the Schwarzschild radius.

On the contrary, a free falling Lemaitre clock moves from some radius r_1 to a smaller radius r_2—which can well be the gravitational radius or even the origin—within finite τ-time $\Delta\tau_{12}$. Indeed, setting $d\rho = 0$ in (5) gives

$$\Delta\tau_{12} = -\int_{r_1}^{r_2}\sqrt{\frac{r}{r_g}}dr = \frac{2}{3}\left(\frac{r_1^{3/2} - r_2^{3/2}}{r_g^{1/2}}\right) \, . \tag{8}$$

Event horizons and black holes

Along the trajectory of a radial light ray

$$ds^2 = d\tau^2 - \frac{r_g}{r}d\rho^2 = 0 \, , \tag{9}$$

which gives

$$d\rho = \pm\sqrt{\frac{r}{r_g}}d\tau \, , \tag{10}$$

where plus and minus describe the rays of light sent correspondingly up and down.

Isolating $d\rho$ in (5) and inserting the result into (10) shows that along the trajectory

$$dr = \left(\pm 1 - \sqrt{\frac{r_g}{r}}\right)d\tau \, . \tag{11}$$

Apparently if $r < r_g$ then there is always $dr < 0$ and thus the light rays emitted radially inwards and outwards both end up at the origin. In other words no signal can escape from inside the gravitational radius. This phenomenon is called the event horizon.

Therefore a massive object with a size less than the gravitational radius, called a black hole, is completely under the event horizon and its interior is totally invisible.

The trajectories of massive bodies and light rays inside the gravitational radius both end up in the origin where they cannot be extended any further.

The black holes can possibly be detected through their interaction with the matter outside the event horizon.

Exercises

1. Show that in a *synchronous reference frame where the metric has the form* $ds^2 = d\tau^2 + g_{\alpha\beta}dx^\alpha dx^\beta$, *where* $\alpha, \beta = 1, 2, 3$, *the time lines are geodesics.*

2. *Derive the equation of motion (6),*

$$dt = -\sqrt{\frac{r}{r_g}}\frac{1}{(1-\frac{r_g}{r})}dr \, ,$$

for a body in a radial free fall from infinity towards a black hole using the geodesic equations in the Schwartzschild metric.

Hints: find dt/ds from the t geodesic equation; find dr/ds from the r geodesic equation eliminating $t(s)$ using the expression for the metric and integrating once; divide the two. [3]

3. *In the Schwarzschild metric a body is falling free radially toward the center. What is its coordinate velocity dr/dt at radius r? What is its locally measured velocity at the same place? Hints: in the Schwarzschild metric the locally measured radial length is given as $d\hat{r}^2 = (1-\frac{2M}{r})^{-1}dr^2$ and the locally measured time is given as $d\hat{t}^2 = (1-\frac{2M}{r})dt^2$.*

4. *A radio transmitter is free falling radially toward a black hole. When the transmitter is approching the gravitational radius an outside observer measures its radio signal to be red-shifted as $\omega = \omega_0 e^{-\lambda t}$. Estimate the mass of the black hole from the measured λ. Hints: $\omega_0 = \omega/\sqrt{g_{00}}$; $r - r_g = (r_1 - r_g)e^{-(t-t_1)/r_g}$ (see equation (7)).*

5. *Calculate the proper time it takes for a Lemaitre clock to fall from the gravitational radius to the center of a black hole. For a black hole with the solar mass specify this time in seconds.*

6. *Show that in Newtonian mechanics the circular planetary orbits around stars are stable against small radial perturbations. Show that in general relativity circular orbits are stable only if $r > 6M$. Hints: consider a circular orbit with a small perturbaion, $u = u_0 + \delta u$;*

[3] The radial geodesic equation, $\frac{du_r}{ds} = \frac{1}{2}g_{bc,r}u^b u^c$, for the radial motion ($u^\theta = u^\phi = 0$) in the Schwartzschild metric is given as

$$\frac{d}{ds}\left(-\frac{1}{1-\frac{R}{r}}\dot{r}\right) = \frac{1}{2}\left(1-\frac{R}{r}\right)_{,r}\dot{t}^2 + \frac{1}{2}\left(-\frac{1}{1-\frac{R}{r}}\right)_{,r}\dot{r}^2 , \tag{12}$$

where $\dot{}$ denotes $\frac{d}{ds}$ and $R \equiv r_g$ is the gravitational radius of the central body. We can eliminate $\dot{t}^2$ from the equation using the expression for the metric for the radial motion,

$$ds^2 = \left(1-\frac{R}{r}\right)dt^2 - \frac{1}{1-\frac{R}{r}}dr^2 , \tag{13}$$

or, equivalently,

$$\dot{t}^2 = \frac{1+\frac{1}{1-\frac{R}{r}}\dot{r}^2}{1-\frac{R}{r}} . \tag{14}$$

Substituting this into the radial equation gives (after few identical transformations),

$$\ddot{r} = -\frac{1}{2}\frac{R}{r^2} , \tag{15}$$

which formally is the Newtonian equation. It can be integrated once by multiplying with $\dot{r}$, which gives

$$\frac{d}{ds}\dot{r}^2 = \frac{d}{ds}\left(\frac{R}{r}\right) , \tag{16}$$

and (explain)

$$\frac{dr}{ds} = -\sqrt{\frac{R}{r}} . \tag{17}$$

Now, the t-equation reads

$$\frac{d}{ds}\left(\left(1-\frac{R}{r}\right)\dot{t}\right) = 0 , \tag{18}$$

with the first integral

$$\left(1-\frac{R}{r}\right)\dot{t} = E = 1 , \tag{19}$$

which gives

$$\frac{dt}{ds} = \frac{1}{1-\frac{R}{r}} . \tag{20}$$

Finally,

$$\frac{dr}{dt} = \frac{(\frac{dr}{ds})}{(\frac{dt}{ds})} = -\sqrt{\frac{R}{r}}\left(1-\frac{R}{r}\right) . \tag{21}$$

derive the equation for δu in the lowest order; investigate whether the perturbation remains small or increases.

7. *Show that equatorial orbits in the Schwarzschild metric are stable. Consider for simplicity a circular orbit. Hint: consider an equatorial orbit with a small perturbation, $\theta = \pi/2 + \delta\theta$; derive the equation for $\delta\theta$ in the lowest order; show that the perturbation remains small; interpret the solution.*