Motion of free bodies in gravitational fields

In general relativity free test bodies—that is, bodies affected only by inertial and gravitational forces—moves along geodesics. Massive bodies distort space-time in their vicinity causing the geodesics to become curved.

Geodesics

In special relativity free bodies move along straight lines, that is, the curves with extremal measure[1]. This postulate is generally covariant and can be directly applied to curved spaces, and, consequently, due to equivalence principle, to gravitational fields. Therefore in general relativity free bodies[2] also move along curves with extremal measure.

A curve with extremal measure is called a *geodesic*. Geodesic is a generalization of the notion of a *line* to curved spaces. The term comes from *geodesy*, the science of measuring the size and shape of Earth. In the original sense, a geodesic was the shortest route between two points on the Earth's surface, namely, a segment of a great circle. The term has since been generalized to include measurements in more general mathematical spaces.

Importantly, the postulate of free motion along geodesics can be formulated as a variational principle: the variation of the measure vanishes on the curve actually taken by the free body,

$$\delta \int ds = 0 \,. \tag{1}$$

Geodesic as constant velocity trajectory

Alternatively (but equivalently, of course) one can postulate that in special relativity a free body moves with constant velocity,

$$du^a = 0 \,, \tag{2}$$

where the velocity vector u^a of a moving body is defined as

$$u^a = \frac{dx^a}{ds} \,, \tag{3}$$

where dx^a is the infinitesimal movement of the body along the trajectory and $ds = \sqrt{g_{ab}dx^a dx^b}$ is the invariant interval (the metric).

Apparently the velocity u^a is a vector and can be directly used in curved spaces as a generally covariant quantity. However the differential du^a in equation (2) is not generally covariant and cannot by used in the curved space of general relativity. A suitable generalization to curved spaces would be to substitute the normal differential du^a with the covariant differential Du^a,

$$Du^a = 0 \,. \tag{4}$$

This equation is generally covariant and is actually the *geodesic equation*. It can be also written as

$$\frac{du^a}{ds} + \Gamma^a_{bc} u^b u^c = 0 \,, \tag{5}$$

or

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} = 0 \,. \tag{6}$$

One can interpret this equation as the relativistic generalization of Newton's second law of motion: acceleration of the body equals the force acting on the body (divided by mass). Therefore the quantities $(-)\Gamma^a_{bc} u^b u^c$ appear as forces—inertial and gravitational—acting on the body. Since the Christoffel symbols (and hence the forces) are proportional to derivatives of the metric tensor, one can say that the latter plays the role of the potential of gravitational forces. Thus the gravitational field is given by the metric tensor of the time-space and is therefore a tensor field.

[1] The measure of a curve is given by the integral $\int ds$, where ds is the metric, taken along the curve.

[2] Remember though that in general relativity a free body is a body free from the influence of *physical forces*; gravitational forces are equivalent to inertial forces and do not count as *physical* forces.

Geodesic as extremal trajectory

The measure μ of a trajectory (of a moving body) is defined as the sum of infinitesimal intervals ds along the trajectory,

$$\mu = \int ds\,. \tag{7}$$

The extremal trajectory is the one where the variation of the measure as function of the trajectory vanishes,

$$\delta\mu = 0\,. \tag{8}$$

To calculate the variation of the measure we first vary the interval ds,

$$\delta ds = \delta\sqrt{g_{ab}dx^a dx^b} = \frac{1}{2}\frac{1}{\sqrt{g_{ab}dx^a dx^b}}\delta\left(g_{ab}dx^a dx^b\right) = \frac{1}{2}\frac{1}{ds}\left(\delta g_{ab}dx^a dx^b + 2g_{ab}\delta dx^a dx^b\right)\ . \tag{9}$$

Using $\delta g_{ab} = g_{ab,c}\delta x^c$ and $\frac{1}{ds}g_{ab}dx^b = u_a$ gives[3]

$$\delta ds = \frac{1}{2}g_{ab,c}u^a u^b \delta x^c ds + \delta dx^c u_c\ , \tag{10}$$

where in the second term the summation index a was replaced with c.

Assuming the functions are smooth enough we can exchange the order of differentials in the second term and integrate it by parts using

$$\delta dx^c u_c = d\delta x^a u_a = d(\delta x^a u_a) - \delta x^c du_c. \tag{11}$$

The full differential does not contribute to the variation, and we finally arrive at

$$\delta\mu = \int ds\delta x^c\left(-\frac{du_c}{ds} + \frac{1}{2}g_{ab,c}u^a u^b\right) = 0\,. \tag{12}$$

Since the variation δx is arbitrary, it is the expression in parentheses that should be equal zero identically along the trajectory, which gives the following equation for the curve with extremal measure,

$$\frac{du_c}{ds} = \frac{1}{2}g_{ab,c}u^a u^b\,. \tag{13}$$

This equation is equivalent to the no-acceleration equation (4)[4]. Thus the trajectory with extremal measure is also the trajectory along which the covariant differential of the velocity of freely moving body is zero. This is the consequence of the fact that the action S of a free body with mass m is proportional to the measure of its trajectory,

$$S = -mc\int ds\,. \tag{17}$$

[3]

$$f_{,a} \doteq \frac{\partial f}{\partial x^a}\,.$$

[4]Indeed, from

$$0 = Dg_{ab} = dg_{ab} - \Gamma^e{}_{ac}g_{eb}dx^c - \Gamma^e{}_{bc}g_{ae}dx^c = (g_{ab,c} - \Gamma_{bac} - \Gamma_{abc})dx^c \tag{14}$$

it follows that

$$g_{ab,c} = \Gamma_{bac} + \Gamma_{abc}\,. \tag{15}$$

Inserting this into (13) and using the symmetry $u^a u^b = u^b u^a$ gives

$$\frac{du_c}{ds} - \frac{1}{2}g_{ab,c}u^a u^b = \frac{du_c}{ds} - \frac{1}{2}\left(\Gamma_{bac}u^a u^b + \Gamma_{abc}u^a u^b\right) = \frac{du_c}{ds} - \Gamma_{abc}u^a u^b = \frac{Du_c}{ds}\ , \tag{16}$$

which had to be demonstrated.

Trajectories of rays of light

The equation (4) is not applicable to the propagation of a ray of light since the interval ds along the ray is always zero. In this case one has to use certain parameter, λ, which varies (smoothly) along the ray. Then one can introduce the wave-vector, $k^a = dx^a/d\lambda$. In special relativity the ray of light propagates along a line where $dk^a = 0$. In a curved space in analogy with (4) this becomes

$$Dk^a = 0\,, \tag{18}$$

or

$$\frac{dk^a}{d\lambda} + \Gamma^a_{bc}k^b k^c = 0\,. \tag{19}$$

These equations, together with the condition that for the ray of light always $k^a k_a \propto dx^a dx_a = ds^2 = 0$, also determine the parameter λ should one wish to find it.

The trajectories of the rays of light are called *null geodesics*.

Exercises

1. Consider a generalization of the non-relativistic Lagrangian of a free body in flat space, $\mathcal{L} = \frac{1}{2}m\vec{v}^2$, to a curved space with metric $dl^2 = g_{\alpha\beta}x^\alpha x^\beta$,

$$\frac{1}{2}m\vec{v}^2 \to \frac{1}{2}mg_{\alpha\beta}v^\alpha v^\beta \;.$$

Show that the corresponding Euler-Lagrange equation[5] is the geodesic equation for the given space.

2. Consider a further generalization,

$$S = \int \frac{1}{2}mg_{\alpha\beta}v^\alpha v^\beta dt \to S = -m\int \frac{1}{2}g_{ab}u^a u^b ds \;.$$

Show that the corresponding Euler-Lagrange equation is the relativistic geodesic equation.

3. Show that the Euler-Lagrange equation for the action

$$S = -m\int ds = -m\int \frac{ds}{d\lambda}d\lambda = -m\int \sqrt{g_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda}}d\lambda \doteq \int \mathcal{L}\left(x^a, \frac{dx^a}{d\lambda}\right)d\lambda\;,$$

(where λ is a paramter which changes smoothly along the trajectory) is the relativistic geodesic equation[6].

4. Prove that the two postulates,

$$Du_c = 0\;,$$

and

$$\delta\int ds = 0\,,$$

for the motion of free bodies in curved spaces are equivalent[7].

[5] The Euler-Lagrange equation is given as

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = \frac{\partial\mathcal{L}}{\partial x^\mu}\;.$$

[6] Hints: $ds/d\lambda = \mathcal{L}$, $df/d\lambda = (df/ds)(ds/d\lambda) = (df/ds)\mathcal{L}$

[7] That is, pove that (5) and (13) are equivalent.

5. Consider the parametric equations for a line in Cartesian coordinates x and y,

$$\frac{d^2x}{ds^2} = 0\,,\ \frac{d^2y}{ds^2} = 0\,. \tag{20}$$

Make a coordinate transformation to polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$) and derive the corresponding equations in the r, θ coordinates. Prove that they are identical to the geodesic equation (6).

6. Apply the generally covariant equation for the trajectory of a ray of light, $Dk^a/d\lambda = 0$ (where $k^a = dx^a/d\lambda$, where $x^a(\lambda)$ follows the trajectory), to a flat two-dimensional space with polar coordinates and find the corresponding trajectories[8].

7. Find the equations for geodesics on the surface of a sphere. Hint: integrate the ϕ-equation once and then rewrite the θ-equation for the function $u(\phi) = \cot(\theta(\phi))$.[9]

[8] The first of the two geodesic equations,

$$\frac{d}{d\lambda}\left(r^2\frac{d\phi}{d\lambda}\right) = 0\,,\ \text{and}\ \frac{d^2r}{d\lambda^2} = r\left(\frac{d\phi}{d\lambda}\right)^2\,,$$

is easily integrated,

$$r^2\frac{d\phi}{d\lambda} = J\,,$$

where J is the integration constant. The second is traditionally integrated by a substitution, $r(\lambda) = 1/u(\phi(\lambda))$, which transforms the second equation into

$$\frac{d^2u}{d\phi^2} + u = 0$$

with the general solution $u = u_0\sin(\phi - \phi_0)$ which is a general parametrization of a line in polar coordinates.

[9] The geodesic equations are

$$\ddot{\theta} = \sin\theta\cos\theta\dot{\phi}^2\ ,\ \ \frac{d}{dt}\left(\sin^2\theta\dot{\phi}\right) = 0\ ,$$

where dot denotes time-derivative, $\dot{\phi} \,\hat{=}\, d\phi/dt$. The second equation can be easily integrated,

$$\sin^2\theta\dot{\phi} = k\ ,$$

where k is the integration constant. The first equation must be rewritten for the function $\theta(\phi(t))$ instead of $\theta(t)$. The second t-derivative of θ then becomes (using $\dot{\phi} = k/\sin^2\theta$),

$$\ddot{\theta} = -\frac{k^2}{\sin^2\theta}\left(-\frac{\theta''}{\sin^2\theta} + 2\frac{\cos\theta}{\sin^3\theta}\theta'^2\right) = -\frac{k^2}{\sin^2\theta}\left(\cot\theta\right)''\ ,$$

where ptime denotes ϕ-derivative, $\theta' \,\hat{=}\, d\theta/d\phi$. The first geodesic equation then becomes (again using $\dot{\phi} = k/\sin^2\theta$)

$$-(\cot\theta)'' = \cot\theta\ ,$$

with the general solution

$$\cot\theta = A\sin(\phi - \phi_0)\ ,$$

(where A and ϕ_0 are integration constats) which is the equation for a great circle.