Quantum measurement and preparation of Gaussian states

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English summary

Our ability to access and manipulate quantum systems has been rapidly progressing in recent decades. This progress has required new theoretical tools to describe and interpret quantum experiments, and has also caused great interest in the development of promising quantum technologies with practical applications. A major challenge is given by the fact that quantum effects are highly susceptible to decoherence and therefore a quantum system needs to be adequately isolated from its environment, but at the same time it is important to have reliable and precise control of the system's state and dynamics.

This dissertation is focused on Gaussian states of continuous variable systems that are experimentally realized in a variety of setups. The general theory is developed to describe the stochastic evolution of continuously monitored systems and to include a new element of measurement theory that allows for improvements in the precision of state and parameter estimation through the reinterpretation the data extracted from a system through measurement. The theory is applied to give a full characterization of the state of an optomechanical setup that is continuously measured as well as to study the cooling of the system due to its monitored dynamics.

Dansk resumé

Vores evne til at tilgå og manipulere kvantesystemer er blevet forbedret hurtigt i de sidste årtier, hvilket har krævet nye teoretiske værktøjer til at beskrive og fortolke kvantemekaniske eksperimenter, og også medført stor interesse i udviklingen af lovende nye kvanteteknologier med praktiske anvendelser. En stor udfordring kommer af at kvantemekaniske effekter er meget følsomme overfor dekohærens, og kvantemekaniske systemer derfor skal bruge passende isolerede omgivelser, men på samme tid er det vigtigt at have pålidelig og præcis kontrol af systemets tilstand og dynamik.

Denne afhandling fokuserer på Gaussiske tilstande af kontinuerte variable i systemer der er eksperimentielt realiserbare i forskellige udgaver. Den generelle teori er udviklet til at beskrive den stokastiske udvikling af kontinuerligt observerede systemer, og til at inkludere et nyt element af målingsteori som tillader forbedringer i præcisionen af tilstands- og parameterestimering ved genfortolkning af data opsamlet fra et system ved målinger. Teorien er anvendt til at lave en fuld karakterisering, af tilstanden af et optomekanisk system som er kontinuerligt målt, og til at studere kølingen af systemet på grund af den observerede dynamik.

Preface

This dissertation presents the research that I have done during my PhD education at the Department of Physics and Astronomy at Aarhus University, Denmark. The research was carried out between April 2015 and April 2018 under the supervision of Klaus Mølmer, and it was funded by Villum Fonden (within QUSCOPE, Villum Foundation Center of Excellence).

List of Publications

[1] Jinglei Zhang and Klaus Mølmer. *Prediction and retrodiction with continuously monitored Gaussian states*. Phys. Rev. A **96**, 062131 (2017).

[2] Jinglei Zhang and Klaus Mølmer. *Quantum-classical hybrid system, quantum parameter estimation with harmonic oscillators* (2018). In preparation.

[3] Magdalena Szczykulska, Georg Enzian, Jinglei Zhang, Klaus Mølmer, and Michael Vanner. *Cooling of a mechanical resonator by spontaneous Brillouin scattering* (2018). In preparation.

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Chapter 1

Introduction and outline

To study any quantum phenomena there is a delicate trade-off between being able to monitor and control the system, and, at the same time, keeping it isolated from the environment and other decoherence sources that would destroy quantum effects. Despite this, such phenomena have now been observed in a variety of systems ranging from single atoms to macroscopic Bose-Einstein condensates. The concrete possibilities of exploiting quantum systems for technological applications justify the ever-growing interest in the field.

Measurement theory has a fundamental role in the control and preparation of quantum systems. This dissertation aims at developing theoretical tools concerning the study of monitored systems and the optimal way to interpret measurement outcomes.

Chapter 2 presents the theory and focuses on the formal characterization of measurements in quantum mechanics from the basic postulates to the more applicable concept of generalized measurement. These elements are then used to discuss past quantum state theory, which employs Bayesian conditioning on future measurement outcomes to give a better description of a quantum state at a past time.

Chapter 3 is devoted to the theory of continuous variable systems and introduces Gaussian states in details. One of their main advantages is that Gaussian states are both experimentally feasible in many experimental setups, and they can be efficiently characterized with theoretical tools that often allow for analytical results.

Chapter 4 gives the theory of an open quantum system interacting with its environment and being subject to time-continuous probing. Particular attention is given to the description of optical systems that are in such situations

and the implementation and interpretation of relevant optical measurements are discussed.

Chapter 5 contains original work that can be found in Refs. [1, 2]. The concepts that are previously discussed are used to specialize the general past quantum state theory to the case where a Gaussian description is possible and therefore obtain a simple characterization of a past quantum state in Gaussian formalism. The Chapter concludes by applying this theory to some cases and exploring the consequences and features of retrodiction in different scenarios.

Chapter 6 contains the study of an optomechanical setup with optical and acoustical whispering gallery modes of a silica resonator that interact via Brillouin scattering. The content of this Chapter is covered in Ref. [3]. The Gaussian formalism is applied to show how the monitoring of the optical mode allows to give the full characterization of the state of the system conditioned on the measurement. In particular the monitored dynamics is studied and it is show that it reduces the number of excitations in the mechanical mode, therefore improving the performance of standard sideband cooling.

Chapter 2

Measurements in quantum mechanics

The objective of this Chapter is to introduce the basic concepts of quantum mechanics and focus on the fundamentals of quantum measurement theory. While the interpretation of the elements we present has sparked much debate about their precise ontological meaning, this discussion goes beyond the purpose of the present thesis and we will focus on the formal aspect of the description of quantum measurement.

2.1 Projective measurement

To every quantum system is associated a complex Hilbert space $\mathcal{H}$ with inner product $\langle\cdot|\cdot\rangle$. We adopt the braket notation so that a ket vector $|\psi\rangle$ is an element of $\mathcal{H}$, while $\langle\varphi|$ indicates the bra vector in the dual space of $\mathcal{H}$ and the inner product of vectors $|\psi\rangle$ and $|\varphi\rangle$ is simply $\langle\varphi|\psi\rangle$.

The state of the system is described by a vector $|\psi\rangle$ that should be normalized: $\langle\psi|\psi\rangle = 1$.

An observable of the system is a self-adjoint operator on the Hilbert space $\mathcal{H}$. Due to the spectral theorem, any self-adjoint operator $\hat{A}$ admits a spectral decomposition $\hat{A} = \sum_n \lambda_n \hat{\Pi}_n$, where λ_n are the (real) eigenvalues of $\hat{A}$ and $\hat{\Pi}_n$ are the projection operators onto the eigenspaces relative to λ_n and thus satisfy $\hat{\Pi}_n^\dagger = \hat{\Pi}_n$ and $\hat{\Pi}_n^2 = \hat{\Pi}_n$. Furthermore, the projectors are mutually orthogonal, namely $\hat{\Pi}_n\hat{\Pi}_m = \delta_{nm}$ and they are normalized to the identity operator $\sum_n \hat{\Pi}_n = \mathbb{1}$.

When an *observable* $\hat{A}$ is measured on the state $|\psi\rangle$, the measurement postulate states that the possible outcomes of the measurement are the eigenvalues λ_n and that the probability of obtaining λ_n is given by

$$\Pr(\lambda_n) = \langle\psi|\hat{\Pi}_n|\psi\rangle\,. \tag{2.1}$$

It follows trivially from the normalization of the projection operators that $\sum_n \Pr(\lambda_n) = 1$ and from the properties of the inner product that, as one expects from a probability distribution, $\Pr(\lambda_n) > 0$.

After the measurement has produced outcome λ_n the state of the system collapses to

$$|\psi\rangle \mapsto \frac{\hat{\Pi}_n\,|\psi\rangle}{\sqrt{\Pr(\lambda_n)}}. \tag{2.2}$$

When considering a closed system, that is a system that is not interacting with its surroundings, it is postulated that the time evolution of a quantum state is determined by a self-adjoint operator $\hat{H}$, called Hamiltonian, according to Schrödinger's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|\psi(t)\rangle = -i\hat{H}\,|\psi(t)\rangle\,, \tag{2.3}$$

which leads to the solution

$$|\psi(t)\rangle = \hat{U}_t\,|\psi(0)\rangle\,, \tag{2.4}$$

where the unitary operator is $\hat{U}_t = \mathrm{e}^{-i\hat{H}t}$.

If, on the other hand, we consider an open system, then the interaction with the environment will introduce uncertainty about the precise state of the system, and it can not be described by a single vector $|\psi\rangle$. However, similarly to what happens in classical statistical physics, we can describe the state as being in a statistical mixture where it is with probability p_i in the state $|\psi_i\rangle$, for $i = 1, \ldots, n$. If $n = 1$ we are in the case that was previously discussed and we say that a system is in a *pure state*, otherwise we say that it is in a *mixed state*.

In this context a convenient description is provided by the *density operator* (or *density matrix*)

$$\rho = \sum_i p_i\,|\psi_i\rangle\langle\psi_i|\,. \tag{2.5}$$

From the definition of the density operator it follows that any such ρ has the following properties:

1. ρ is a self-adjoint operator;
2. $\rho \geq 0$, that is ρ is a positive semi-definite operator, namely that all its eigenvalues are non-negative;
3. $\mathrm{Tr}\,[\rho] = 1$, since $\{p_i\}$ is a set of probabilities and therefore $\sum_i p_i = 1$.

It is also easy to see that the set of density matrices is a convex set, that is, convex linear combination of density matrices is still a density matrix, consistently with its probabilistic interpretation.

When an observable $\hat{A}$ is measured then it is possible to prove from the measurement postulate that the probability of outcome λ_n is

$$\Pr(\lambda_n) = \mathrm{Tr}\left[\hat{\Pi}_n \rho\right], \tag{2.6}$$

and the density matrix conditioned on the outcome is

$$\rho \mapsto \frac{\hat{\Pi}_n \rho \hat{\Pi}_n}{\Pr(\lambda_n)}. \tag{2.7}$$

On the other hand, the unitary evolution induced by the Hamiltonian can easily be found to be

$$\rho(t) = \hat{U}_t \rho(0) \hat{U}_t^\dagger. \tag{2.8}$$

2.2 Generalized measurements

It is almost always the case that we do not practically have access to the system directly, which instead is interacting with a *meter* (or *environment*), and it is by observing the effect of the system on the meter that we can deduce information about the former. This scheme is called a *generalized measurement*.

In more precise terms, consider a system and a meter supposed initially to be in the separable state $|\psi\rangle_S |\theta\rangle_E$. Let them evolve for some time through a global unitary evolution $\hat{U}$ that couples the system with the meter and gives the state $\hat{U}(|\psi\rangle_S |\theta\rangle_E)$. We now measure on the meter the observable $\hat{A}_E = \sum_n \lambda_n |\lambda_n\rangle_E \langle\lambda_n|$, where we have assumed for simplicity that the eigenvalues of the observable are all non-degenerate and therefore the projectors are only rank-1 projectors $|\lambda_n\rangle_E \langle\lambda_n|$. This means applying Born's rule on the composite system with the operator $\mathbb{1}_S \otimes |\lambda_n\rangle_E \langle\lambda_n|$.

The probability of obtaining an outcome λ_n is

$$\Pr(\lambda_n) = {}_S\langle\psi|\,{}_E\langle\varphi|\,\hat{U}^\dagger \left(\mathbb{1}_S \otimes |\lambda_n\rangle_E \langle\lambda_n|\right) \hat{U}\,|\psi\rangle_S\,|\varphi\rangle_E \tag{2.9}$$

and the state after the measurement is

$$\begin{aligned}&\frac{\mathbb{1}_S \otimes |\lambda_n\rangle_E \langle\lambda_n| \hat{U} |\psi\rangle_S |\theta\rangle_E}{\sqrt{\Pr(\lambda_n)}} \\ =&\frac{\left(\hat{M}_n |\psi\rangle_S\right) \otimes |\lambda_n\rangle_E}{\sqrt{\Pr(\lambda_n)}},\end{aligned} \tag{2.10}$$

where we have defined the linear operator $\hat{M}_n$ acting only on the system's Hilbert space as

$$\hat{M}_n |\psi\rangle_S = {}_E\langle\lambda_n| \hat{U} \left(|\psi\rangle_S |\theta\rangle_E\right). \tag{2.11}$$

By generalizing to the case where the system is initially in a mixed state ρ we have that the probability distribution for the outcome is

$$\Pr(\lambda_n) = \mathrm{Tr}\left[\hat{M}_n \rho \hat{M}_n^\dagger\right], \tag{2.12}$$

and the collapse of the state follows

$$\rho \mapsto \frac{\hat{M}_n \rho \hat{M}_n^\dagger}{\Pr(\lambda_n)}. \tag{2.13}$$

This conditional state is similar to what we had in Eq. (2.7), where the projectors $\hat{\Pi}_n$ have been substituted by the operators $\hat{M}_n$.

The operator $\hat{E}_n = \hat{M}_n^\dagger \hat{M}_n$ is called *probability operator* or *effect* for the outcome n and determines completely the probability distribution.

The operators $\hat{M}_n$ are called *measurement operators* or *detection operators* and the set $\{\hat{M}_n\}$ defines a so-called positive-operator valued measure (POVM). It can be proven that the measurement operators satisfy the following two properties

$$\sum_n \hat{M}_n^\dagger \hat{M}_n = \mathbb{1}, \tag{2.14}$$

$$\hat{M}_n^\dagger \hat{M}_n \geq 0, \tag{2.15}$$

which guarantee that $\Pr(\lambda_n)$ is a proper probability distribution. Both these properties are satisfied by the projectors $\{\hat{\Pi}_n\}$ derived from a spectral decomposition of an observable. But, in the case of generalized measurements, the effect operators are not necessarily mutually orthogonal.

2.3 Quantum operations

We have studied so far some specific examples of evolution for a quantum state. In this Section we want to characterize the minimum requirements for a map that transforms density operators into density operators that is physically admissible. We call such a map a *quantum operation*.

Consider a map $\rho \mapsto \Phi(\rho)$. First of all, we require that $\Phi(\rho)$ is a *bona fide* density matrix, therefore we need that

1. the map is linear: $\Phi(\alpha\rho_1 + \beta\rho_2) = \alpha\Phi(\rho_1) + \beta\Phi(\rho_2)$;
2. the map is trace-preserving: $\mathrm{Tr}\left[\Phi(\rho)\right] = \mathrm{Tr}\left[\rho\right]$;
3. Φ is positive, i.e. $\Phi(\rho) \geq 0$ if $\rho > 0$ (and it is implied that $\Phi(\rho)$ is self-adjoint).

Additionally, we require also that

4. the map is *completely positive*, which means that if we consider any extension of the original Hilbert space with an ancillary system, then the map $\Phi \otimes \mathbb{1}$ acting on the composite system is positive.

It can be shown that complete positivity implies positivity of a map, but that the inverse is not in general true.

A map that satisfies all the above properties is also called a completely positive and trace preserving map, or CPTP map. The physicality of CPTP maps can also be interpreted through the following theorem [4].

Naimark's extension theorem A map Φ on a system is CPTP if and only if it can be realized by extending the system with an appropriate ancillary system in a initially separable state, letting the composite system evolve unitarily and taking eventually the partial trace over the ancillary system.

By considering the physical representation of a CPTP map, it is possible prove that any quantum operation can be written in Kraus form, namely, that there exists a set of operators $\{\hat{K}_k\}$, called Kraus operators, that satisfy $\sum_k \hat{K}_k^\dagger \hat{K}_k = \mathbb{1}$ and are such that

$$\Phi(\rho) = \sum_k \hat{K}_k \rho \hat{K}_k^\dagger. \tag{2.16}$$

The unitary evolution of a quantum system with Hamiltonian $\hat{H}$ is the simplest case of Kraus representation where there is only one Kraus operator given by the unitary $\hat{U}_t = \mathrm{e}^{-i\hat{H}t}$, and Eq. (2.16) is a generalization

needed when describing an open quantum system that is interacting with an environment.

In the case of generalized measurement we can recover an evolution in Kraus form by considering the *unselective evolution* which corresponds to the idea of performing a measurement but discarding the result. In this case the state of the system after the measurement is given by the statistical average

$$\begin{aligned}\rho_{\text{uns}} &= \sum_n \Pr(\lambda_n)\rho_{\text{conditioned on } n} \\ &= \sum_n \hat{M}_n \rho \hat{M}_n^\dagger.\end{aligned} \tag{2.17}$$

So an unselective evolution is described with Kraus operators corresponding to the generalized measurement operators.

Due to Naimark's theorem we know that CPTP maps describe all possible deterministic physical evolutions of a system, but the back action induced by any measurement is necessarily stochastic. Nevertheless, the state conditioned on one specific measurement result given by Born's rule in Eq. (2.13) can be seen as the evolution induced by only the Kraus operator corresponding to the measurement result that was obtained, $\rho \mapsto \hat{M}_n \rho \hat{M}_n^\dagger$, and which has been normalized to have a proper density operator with unit trace, $\rho \mapsto \hat{M}_n \rho \hat{M}_n^\dagger / \operatorname{Tr}\left[\hat{M}_n \rho \hat{M}_n^\dagger\right]$.

To conclude, the Kraus representation allows us to describe not only the evolution of an open quantum system, but also the measurement back action when a generalized measurement is performed and to put these two processes on the same level.

2.3.1 Dual maps

Given a CPTP map Φ, the corresponding *dual map* is defined as the map Φ^* such that

$$\operatorname{Tr}\left[\Phi(\hat{A})\hat{B}\right] = \operatorname{Tr}\left[\hat{A}\Phi^*(\hat{B})\right] \tag{2.18}$$

for all operators $\hat{A}$ and $\hat{B}$.

It can be shown that the dual map is always unique, linear, completely positive and unital, that is $\Phi^*(\mathbb{1}) = \mathbb{1}$, but in general not trace-preserving.

The use of CPTP maps belongs to Schrödinger's picture, since they describe the evolution of the state of a quantum system. Dual maps arise naturally when going to Heisenberg picture and they give the corresponding evolution for the observables of the system.

The dual map of a CPTP map in Kraus representation given in Eq. (2.16) can be easily found by considering

$$\mathrm{Tr}\left[\Phi(\rho)\hat{A}\right] = \sum_k \mathrm{Tr}\left[\hat{K}_k \rho \hat{K}_k^\dagger \hat{A}\right] = \mathrm{Tr}\left[\rho \sum_k \hat{K}_k^\dagger \hat{A} \hat{K}_k\right], \tag{2.19}$$

and therefore

$$\Phi^*(\hat{A}) = \sum_k \hat{K}_k^\dagger \hat{A} \hat{K}_k. \tag{2.20}$$

2.4 Past quantum state theory

We have described so far the standard treatment of measurement and conditional states in quantum mechanics. At every time the state of the system is given by a density operator ρ conditioned on all previous measurements and it provides the probability distribution for any measurement via Born's rule. For this reason we can consider this a theory that goes forward in time and bases predictions at every time on the information acquired in the past.

In this Section we present the past quantum state formalism introduced first in Ref. [5] and illustrate the derivation and the idea of past quantum state.

Consider a system that is probed for some finite time interval. While for some intermediate time t the standard quantum state $\rho(t)$ provides only information for times before t, a past quantum state consists of a couple $(\rho(t), E(t))$, where $E(t)$ is called *effect matrix* and is conditioned on measurements performed at times later than t. The past quantum state allows therefore to employ the information acquired about the system during the whole interval for a state or parameter estimation task at an intermediate time.

Let us consider a system that is continuously monitored for a time interval $[0, T]$ with initial density operator ρ_0. Let us divide the time interval in N infinitesimal steps of size $\mathrm{d}t$ that we call $\{t_i\}_{i=0,\dots,N}$. To keep a simpler notation we assume that at every time the evolution of the system is described by measurement operators $\{\hat{M}_n\}$. The following discussion, however, can be easily generalized to the case where different measurement (or Kraus) operators are applied at different times.

If $\{n_0, \dots, n_N\}$ is the measurement record, i.e. the specific measurement outcome obtained at every t_i, then at every time the state of the system is determined by

$$\rho(t_0) = \rho_0, \qquad \rho(t_{i+1}) \propto \hat{M}_{n_i}\, \rho(t_i)\, \hat{M}_{n_i}^\dagger. \tag{2.21}$$

If at time t_k we want to predict the probability distribution for a measurement $\{\hat{\Omega}_m\}$, then the probability of obtaining outcome m conditioned on the measurement record up to that time is given by

$$\begin{aligned}\Pr(m|n_0,\dots,n_k) &= \frac{\mathrm{Tr}\left[\hat{\Omega}_m\rho(t_k)\hat{\Omega}_m^\dagger\right]}{\sum_m' \mathrm{Tr}\left[\hat{\Omega}_{m'}\rho(t_k)\hat{\Omega}_{m'}^\dagger\right]} \\ &= \frac{\mathrm{Tr}\left[\hat{\Omega}_m\hat{M}_{n_{k-1}}\cdots\rho_0\cdots\hat{M}_{n_{k-1}}^\dagger\hat{\Omega}_m^\dagger\right]}{\sum_m' \mathrm{Tr}\left[\hat{\Omega}_{m'}\rho(t_k)\hat{\Omega}_{m'}^\dagger\right]}.\end{aligned} \tag{2.22}$$

However, an observer that has access to the full measurement record can ask what the probability of obtaining outcome m is, conditioned on the full measurement record. We call this a *past probability distribution* and we indicate it with the notation $\Pr_{\mathrm{p}}(m,t_k) \equiv \Pr(m,t_k|n_0,\dots,n_N)$. It can be calculated using the conditional probability formula $\Pr(A|B) = \frac{\Pr(A\cap B)}{\Pr(B)}$, where the events are A ="outcome m at t_k" and B ="obtaining the measurement record $r_1,\dots,r_N$".

By neglecting a normalization constant we get

$$\begin{aligned}\Pr_{\mathrm{p}}(m,t_k) &\propto \Pr(n_0,\dots,n_{k-1},m,n_k,\dots,n_N) \\ &= \mathrm{Tr}\left[\hat{M}_{n_N}\cdots\hat{\Omega}_m\cdots\hat{M}_{n_0}\,\rho_0\,\hat{M}_{n_0}^\dagger\cdots\hat{\Omega}_m^\dagger\cdots\hat{M}_{n_N}^\dagger\right] \\ &= \mathrm{Tr}\left[\hat{\Omega}_m\,\rho(t_k)\,\hat{\Omega}_m^\dagger\,\hat{M}_{n_k}^\dagger\cdots\hat{M}_{n_N}^\dagger\hat{M}_{n_N}\cdots\hat{M}_{n_k}\right],\end{aligned} \tag{2.23}$$

where we have used the cyclic property of the trace to separate the contribution of measurement operators before and after t_k.

We can now define the *effect matrix* as

$$E(t_k) \equiv \hat{M}_{n_k}^\dagger\cdots\hat{M}_{n_N}^\dagger\,\mathbb{1}\,\hat{M}_{n_N}\cdots\hat{M}_{n_k}, \tag{2.24}$$

so that the past probability distribution can be written as

$$\Pr_{\mathrm{p}}(m,t_k) \propto \mathrm{Tr}\left[\hat{\Omega}_m\,\rho(t_k)\,\hat{\Omega}_m^\dagger\,E(t_k)\right]. \tag{2.25}$$

If we include the correct normalization then we obtain that the probability of outcome m conditioned on the measurements both before and after an intermediate time t is

$$\Pr_{\mathrm{p}}(m,t) = \frac{\mathrm{Tr}\left[\hat{\Omega}_m\rho(t)\hat{\Omega}_m^\dagger E(t)\right]}{\sum_{m'}\mathrm{Tr}\left[\hat{\Omega}_{m'}\rho(t)\hat{\Omega}_{m'}^\dagger E(t)\right]}. \tag{2.26}$$

We can see from the definition of the effect matrix in Eq. (2.24) that $E(t)$ can be interpreted as an operator evolving *backwards* in time, starting from the condition of being the identity at the final time. The analogue of the measurement back action on ρ given by Eq. (2.21) for the evolution of E is given by

$$E(t_{N+1}) = \mathbb{1}, \qquad E(t_{i-1}) = \hat{M}^{\dagger}_{n_{i-1}}\, E(t_i)\, \hat{M}_{n_{i-1}}, \tag{2.27}$$

and we can see that this evolution is the dual map of the forward evolution of ρ.

Furthermore, we can observe that, while the density matrix $\rho(t)$ depends on only the measurements done *before* t, the effect matrix $E(t)$ is dependent only on the *later* measurements. In particular, if the system is not probed after t and no further information is acquired after that, then the effect matrix is the identity in the whole interval $[t, T]$, and the past probability distribution in Eq. (2.26) reduces to the standard probability distribution in Eq. (2.12).

The past quantum state formalism has been used to improve parameter estimations [6–8] and to reinterpret temporal correlations in measurement records [9–13]. It has also been applied to experimental setups such as probing of the photon number in a cavity by transmission of atoms [14] and to retrodict past measurements in a superconducting qubit setup [15–18].

Aharonov-Bergmann-Lebowitz rule

The idea of retrodiction was first introduced in Ref. [19], where the authors discuss the irreversibility of the measurement process and give a time-symmetric measurement probability by introducing a system ensemble that is pre- and postselected. Their discussion, however, was limited to the case of projective measurements; we present in the following a simplified version of their discussion following Ref. [20].

Consider a system that has three observables $\hat{A}$, $\hat{B}$, and $\hat{C}$, let us assume for simplicity that their spectra are discrete and non-degenerate, and in particular we have the spectral decomposition $\hat{C} = \sum_i c_i\, |c_i\rangle\langle c_i|$. We preselect the system by measuring observable $\hat{A}$ so that the initial state is an eigenstate $|a\rangle$. At a later time observable $\hat{C}$ is measured and outcome c_i is obtained with probability $\Pr(c_i|a) = |\langle c_i|a\rangle|^2$. Finally, we postselect by measuring $\hat{B}$ and projecting the system into an pure state $|b\rangle$ with probability $\Pr(b|c_i) = |\langle b|c_i\rangle|^2$. The retrodiction task is to give the probability distribution for the intermediate outcome conditioned on the pre- and postselection, and it can be written as

$$\Pr(c_i|(a \cap b)) = \frac{\Pr((c_i \cap b)|a)}{\Pr(b|a)}. \tag{2.28}$$

The numerator is the probability of obtaining outcomes c_i and b conditioned on the preparation of the system in $|a\rangle$, and is therefore $|\langle b|c_i\rangle \langle c_i|a\rangle|^2$, while the numerator is the probability of obtaining b during the final measurement, irrespective of the outcome of the middle measurement, and is therefore $\sum_{i'} |\langle b|c_{i'}\rangle \langle c_{i'}|a\rangle|^2$. The retrodicted probability for c_i with pre- and post-selection is therefore

$$\Pr(c_i|(a \cap b)) = \frac{|\langle b|c_i\rangle \langle c_i|a\rangle|^2}{\sum_{i'} |\langle b|c_{i'}\rangle \langle c_{i'}|a\rangle|^2}. \tag{2.29}$$

Equation (2.29) is called *Aharonov-Bergmann-Lebowitz (ABL) rule*, and it gives a probability distribution for a measurement outcome that is time symmetric, that is, symmetric respect to the inversion of pre- and post-selection.

The ABL rule can be seen simply as a special case of the past probability distribution in Eq. (2.26) where there is only a two-element measurement record and only projective measurements are considered. The middle measurement operator is $\hat{\Omega}_{c_i}$. The time-symmetry that was originally observed is not true in the general case. In fact the past probability is symmetric with respect to the exchange of ρ and E only if the middle measurement operator satisfies $\hat{\Omega}_m^\dagger = \hat{\Omega}_m$.

Chapter 3

Continuous variable systems

The purpose of this Chapter is to introduce the fundamental elements of the theory of quantum continuous variable systems. By continuous variable we mean a system with observables with a continuous spectrum. A prototypical example is a quantum harmonic oscillator, which is completely described by the non-commuting position and momentum operators. In the rest of this thesis we focus on two types of continuous variable systems that allow for the same type of description: light modes and mechanical oscillators. After presenting the theory for these two systems, we go into the general theory for continuous variable systems and introduce the Wigner function and phase space formalism.

3.1 Mechanical and optical resonators

The free Hamiltonian for a one-dimensional mechanical oscillator of mass m and frequency ω is

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{m}{2}\omega^2\hat{Q}^2, \tag{3.1}$$

where $\hat{Q}$ and $\hat{P}$ are the dimensional position and momentum operators that obey the commutation relation $\left[\hat{Q}, \hat{P}\right] = i\hbar$. We introduce the adimensional operators by considering the Hamiltonian

$$\frac{\hat{H}}{\hbar} = \frac{\omega}{2}\left(\frac{\hat{P}^2}{\hbar m\omega} + \frac{m\omega\hat{Q}^2}{\hbar}\right) = \frac{\omega}{2}\left(\hat{p}^2 + \hat{q}^2\right), \tag{3.2}$$

where the adimensional operators are defined as

$$\hat{q} = \sqrt{\frac{m\omega}{\hbar}}\hat{Q}, \qquad \hat{p} = \frac{\hat{P}}{\sqrt{\hbar m\omega}}, \tag{3.3}$$

so that the commutation relation of the new operators is $[\hat{q}, \hat{p}] = i$. We define the zero-point fluctuation amplitude that corresponds to the ground state energy as $x_0 = \sqrt{\frac{\hbar}{2m\omega}}$. The ladder operators for the mechanical mode are then defined by

$$\hat{Q} = x_0(\hat{a} + \hat{a}^\dagger), \qquad \hat{P} = -im\omega x_0(\hat{a} - \hat{a}^\dagger). \tag{3.4}$$

A Fabri-Pérot cavity is made of two mirrors facing each other that sustain a sequence of standing light waves with equally spaced frequencies. The electromagnetic modes that are allowed inside the cavity can be found by solving Maxwell's equation without sources and appropriate boundary conditions. Suppose that the mirrors are aligned in the z direction. Let us consider for simplicity a solution with polarization in the x direction, that is, a specific mode characterized by

$$E_x(z,t) = \sqrt{\frac{2\omega^2}{V\epsilon_0}}q(t)\sin(kz), \tag{3.5}$$

$$B_y(z,t) = \frac{\mu_0\epsilon_0}{k}\sqrt{\frac{2\omega^2}{V\epsilon_0}}\dot{q}(t)\cos(kz), \tag{3.6}$$

where q has the dimensions of a length, V is the effective volume of the cavity, the wave number depends on the frequency according to $k = \frac{\omega}{c}$, and the allowed frequencies are $\omega = \frac{c\pi}{L}n$, with n any integer. For the quantization of a general electromagnetic field we refer to Ref. [21].

The energy of the field can be calculated to be

$$H = \frac{1}{2}\int \mathrm{d}V \left(\epsilon_0 E_x^2 + \frac{1}{\mu_0}B_y^2\right) = \frac{1}{2}\left(\omega^2 q(t)^2 + p(t)^2\right). \tag{3.7}$$

This can be recognized as the Hamiltonian of a harmonic oscillator with unit mass and frequency ω, with canonical variables $q(t)$ and $p(t) = \dot{q}(t)$.

We quantize the electromagnetic field by promoting q and p (and therefore the electric and magnetic field) to operators, and we require that they satisfy the canonical commutation relation $[\hat{q}, \hat{p}] = i\hbar$. The annihilation and creation operators can be defined in the same way as previously done for the

mechanical oscillator, and they are related to the electric and magnetic field by

$$\hat{E}_x(z,t) = \mathcal{E}_0(\hat{a}\,\mathrm{e}^{-i\omega t} + \hat{a}^\dagger\,\mathrm{e}^{i\omega t})\sin(kz), \tag{3.8}$$

$$\hat{B}_y(z,t) = -i\mathcal{B}_0(\hat{a}\,\mathrm{e}^{-i\omega t} - \hat{a}^\dagger\,\mathrm{e}^{-i\omega t})\cos(kz), \tag{3.9}$$

where the dimensional factors are $\mathcal{E}_0 = \sqrt{\frac{\hbar\omega}{V\epsilon_0}}$ and $\mathcal{B}_0 = \frac{\mu_0}{k}\sqrt{\frac{\epsilon_0\hbar\omega^3}{V}}$.

The light inside the cavity can then be described by a superposition of the excitations for the different modes, each characterized by a different frequency $\omega_n = \frac{c\pi}{L}n$. An other common situation in many optics experiments is a propagating beam of light, for example in an optical fiber or more general waveguides. We briefly summarize the theoretical treatment of this situation following Ref. [22].

To model a propagating wave along the z direction we require boundary conditions along the z direction so that the allowed frequencies are an integer multiple of $\Delta\omega = \frac{2\pi c}{L}$. To each frequency we can associate an annihilation operator $\hat{a}_i$ with i integer and such that they obey the bosonic commutation relation $\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij}$. In the limit of $L \to \infty$ the spectrum becomes continuous and it is convenient to define the annihilation operators as function of frequency as

$$\hat{a}(\omega) = \frac{1}{\sqrt{\Delta\omega}}\hat{a}_i, \tag{3.10}$$

so that they obey the commutation relation

$$\left[\hat{a}(\omega), \hat{a}(\omega')\right] = \delta(\omega - \omega'). \tag{3.11}$$

The electromagnetic field propagating in the z direction for a cross section of area A can be found to be

$$\hat{E}_x(z,t) = \hat{E}^+(z,t) + \hat{E}^-(z,t), \tag{3.12}$$

$$\hat{B}_y(z,t) = \hat{B}^+(z,t) + \hat{B}^-(z,t), \tag{3.13}$$

where we have assumed x polarization for simplicity. The $+$ operators denote the annihilation operators' contribution and the $-$ operators are their the Hermitian conjugate, namely

$$\hat{E}^+(z,t) = i\sqrt{\frac{\hbar\omega}{4\pi\epsilon_0 c A}}\int_0^\infty \mathrm{d}\omega\,\hat{a}(\omega)\,\mathrm{e}^{-i\omega(t-z/c)}, \tag{3.14}$$

$$\hat{B}^+(z,t) = i\sqrt{\frac{\hbar\omega}{4\pi\epsilon_0 c^3 A}}\int_0^\infty \mathrm{d}\omega\,\hat{a}(\omega)\,\mathrm{e}^{-i\omega(t-z/c)}, \tag{3.15}$$

Since we are describing an electromagnetic plane wave propagating in the z direction, we can define a normally ordered Poynting vector operator $\hat{S}(z,t)$ in analogy with the classical case as

$$\hat{\boldsymbol{S}}(z,t) = \frac{1}{\mu_0}\left(\hat{\boldsymbol{E}}^-(z,t)\times\hat{\boldsymbol{B}}^+(z,t) - \hat{\boldsymbol{B}}^-(z,t)\times\hat{\boldsymbol{E}}^+(z,t)\right). \tag{3.16}$$

This gives the intensity of the energy flowing through a section of the waveguide as

$$\int_{-\infty}^{\infty} \mathrm{d}t\, \hat{S}(z,t) = \frac{1}{A}\int_0^{\infty} \mathrm{d}\omega\, \hbar\omega \hat{a}^\dagger(\omega)\hat{a}(\omega). \tag{3.17}$$

3.2 Quantum harmonic oscillators

Let us consider n quantum oscillators, each described by its position and momentum operators $\hat{q}_j$, $\hat{p}_j$. As discussed in the previous Section, both mechanical oscillators and light fields can be described with this formalism, but can actually be applied to many other physical systems such as atomic ensembles [23, 24], phonons, optomechanical oscillators [25, 26], trapped ions [27], and BEC oscillation modes [28].

Let us define the vector of quadrature operators $\hat{\boldsymbol{r}}^\intercal = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_n, \hat{p}_n)$. Since every entry of the vector is a Hermitian operator, note that $\hat{\boldsymbol{r}}^\dagger = \hat{\boldsymbol{r}}^\intercal$. We assume bosonic modes so that the commutation relations are

$$\left[\hat{q}_j, \hat{p}_j\right] = i\delta_{jk}, \qquad \left[\hat{q}_j, \hat{q}_k\right] = \left[\hat{p}_j, \hat{p}_k\right] = 0, \tag{3.18}$$

where we choose units with $\hbar = 1$.

Let us for notational convenience define the symplectic matrix Ω with elements defined by $\left[\hat{r}_j, \hat{r}_k\right] = i\Omega_{jk}$. More explicitly

$$\Omega = \bigoplus_{j=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.19}$$

It is easy to check that matrix Ω is orthogonal, since $\Omega^\intercal\Omega = \mathbb{1}$, and also that $\Omega^\intercal = -\Omega$.

Harmonic oscillators are also often described by their ladder operators $\hat{a}_j$, $\hat{a}_j^\dagger$, defined by

$$\hat{a}_j = \frac{\hat{q}_j + i\hat{p}_j}{\sqrt{2}}, \tag{3.20}$$

so that their commutation relation is $\left[\hat{a}_j, \hat{a}_k^\dagger\right] = \delta_{jk}$. The inverse relation is given by

$$\hat{q}_j = \frac{\hat{a}_j + \hat{a}_j^\dagger}{\sqrt{2}}, \qquad \hat{p}_j = \frac{\hat{a}_j - \hat{a}_j^\dagger}{i\sqrt{2}}. \tag{3.21}$$

If we define the number operator $\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$, then the free Hamiltonian for a set of harmonic oscillators can be written as

$$\hat{H} = \sum_{j=1}^{n} \omega_j \left(\hat{a}_j^\dagger \hat{a}_j + \frac{1}{2}\right) = \sum_{j=1}^{n} \omega_j \left(\hat{n}_j + \frac{1}{2}\right). \tag{3.22}$$

A typical basis that is used for harmonic oscillators is the number basis (or Fock basis) which is obtained from the spectral decomposition of the number operator.

Given any quantum state ρ for n bosonic modes, we define the $2n \times 1$ vector of first moments and the $2n \times 2n$ covariance matrix respectively as

$$\langle \boldsymbol{r} \rangle = \mathrm{Tr}\left[\rho \hat{\boldsymbol{r}}\right], \qquad [\sigma]_{jk} = \sigma_{jk} \equiv \langle \{r_j, r_k\} \rangle - 2\langle r_j \rangle \langle r_k \rangle, \tag{3.23}$$

where we use the notation $\langle O \rangle = \mathrm{Tr}\left[\rho \hat{O}\right]$.

It is easy to see that the covariance matrix is real and symmetric, and it can be shown that any real and symmetric matrix corresponds to a quantum state only if Robertson-Schrödinger uncertainty relation is satisfied:

$$\sigma + i\Omega \geq 0, \tag{3.24}$$

that is if the (Hermitian) matrix $\sigma + i\Omega$ has only non-negative eigenvalues. This inequality implies in essence Heisenberg's uncertainty principle.

Let us consider for simplicity one single mode, and define the displacement operator parametrized by a complex number α as

$$\hat{D}(\alpha) = \exp\left[\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right]. \tag{3.25}$$

Since the exponent of the operator is anti-Hermitian, the displacement operator results to be unitary , that is $\hat{D}^\dagger(\alpha)\hat{D}(\alpha) = \mathbb{1}$. By using the Baker-Campbell-Hausdorff formula (see Eq. (E.2)) one can prove that

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha + \beta)\, \mathrm{e}^{i\,\mathrm{Im}[\alpha\beta^*]} = \hat{D}(\alpha + \beta)\, \mathrm{e}^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)}, \tag{3.26}$$

and in particular $\hat{D}^\dagger(\alpha) = \hat{D}(-\alpha)$. The unitary evolution induced on the ladder operators by the displacement operators is given by

$$\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha), \qquad \hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha) = \hat{a}^\dagger + \alpha^*. \tag{3.27}$$

A *coherent state* $|\alpha\rangle$ can be equivalently defined as the state obtained from applying a displacement operator to the vacuum, or equivalently as an eigenvector of the destruction operator. In formulae

$$\hat{D}(\alpha)\,|0\rangle = |\alpha\rangle\,, \qquad \hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,, \tag{3.28}$$

A coherent state in Fock basis is given by

$$|\alpha\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,. \tag{3.29}$$

The set of coherent states is said to be complete, meaning that it gives a resolution of the identity operator

$$\frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha = \mathbb{1}, \tag{3.30}$$

where $\mathrm{d}^2\alpha = \mathrm{d}\,\mathrm{Re}[\alpha]\,\mathrm{d}\,\mathrm{Im}[\alpha]$. Coherent states are not however mutually orthogonal, in fact

$$\langle\beta|\alpha\rangle = \mathrm{e}^{\frac{1}{2}|\alpha-\beta|^2}\,\mathrm{e}^{\frac{1}{2}(\alpha\beta^*-\alpha^*\beta)}\,. \tag{3.31}$$

The completeness property still allows one to express the trace of an operator as

$$\mathrm{Tr}\left[\hat{O}\right] = \frac{1}{\pi}\int_{\mathbb{C}} \langle\alpha|\hat{O}|\alpha\rangle\,\mathrm{d}^2\alpha. \tag{3.32}$$

3.3 Wigner functions and Gaussian states

The idea of associating a phase space quasiprobability distribution to a quantum state is an intriguing parallel to what happens in classical mechanics. This would allow to calculate expectation values of observables as an integral in phase space. Wigner distribution was indeed first introduced in Ref. [29] to study quantum corrections to statistical mechanics. It was later recognized [30, 31] that the Wigner distribution is the function in phase space that provides expectation values of symmetrically ordered operators, while the Glauber-Sudarshan P representation gives the expectation values of normally ordered operators, and the Husimi Q function of anti-normally ordered operators. In the following we will focus on the Wigner function and we will show that it is a powerful tool in characterizing Gaussian states of bosonic systems.

Let us consider n bosonic modes. We can define the displacement operator acting on the global Hilbert space for a vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)^\intercal \in \mathbb{C}^n$ as

$$\hat{D}(\boldsymbol{\alpha}) = \bigotimes_{j=1}^{n} \hat{D}_j(\alpha_j), \tag{3.33}$$

where $\hat{D}_j$ is the displacement operator acting on the j-th mode.

The displacement operator can be also written in terms of the quadrature operators. For example for a single mode

$$\hat{D}(\alpha) = \mathrm{e}^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} = \mathrm{e}^{-i(q\hat{p} - p\hat{q})} = \mathrm{e}^{-i\boldsymbol{r}^\intercal \Omega \hat{\boldsymbol{r}}}, \tag{3.34}$$

where we have defined the vector of cartesian coordinates $\boldsymbol{r}^\intercal = (q, p) = \sqrt{2}(\mathrm{Re}[\alpha], \mathrm{Im}[\alpha])$. We keep the complex variable in the following discussion, for the analogous in terms of real variables we refer to Appendix A.

The set of displacement operators is said to be *complete*, meaning that any operator can be written as a linear combination of displacement operators. If we define the *characteristic function* of an operator $\hat{O}$ as

$$\chi_{\hat{O}}(\boldsymbol{\alpha}) = \mathrm{Tr}\left[\hat{O}\hat{D}(\boldsymbol{\alpha})\right], \tag{3.35}$$

then the Fourier-Weyl relation

$$\hat{O} = \frac{1}{\pi^n} \int_{\mathbb{C}^n} \mathrm{d}^{2n}\boldsymbol{\alpha}\, \chi_{\hat{O}}(\boldsymbol{\alpha}) \hat{D}(-\boldsymbol{\alpha}). \tag{3.36}$$

The Wigner function of an operator $\hat{O}$ is defined as the complex Fourier transform of the characteristic function up to a constant (compare with Eq. (E.6)), in formula

$$W_{\hat{O}}(\boldsymbol{\alpha}) = \frac{1}{\pi^{2n}} \int_{\mathbb{C}^n} \mathrm{d}^{2n}\boldsymbol{\beta}\, \mathrm{e}^{\boldsymbol{\beta}^\dagger \boldsymbol{\alpha} - \boldsymbol{\alpha}^\dagger \boldsymbol{\beta}}\, \chi_{\hat{O}}(\boldsymbol{\beta}). \tag{3.37}$$

It is possible to prove that if the operator $\hat{O}$ is Hermitian, then the Wigner function assumes only real values.

The displacement operators can be proven to be *orthogonal*, that is

$$\mathrm{Tr}\left[\hat{D}(\boldsymbol{\alpha})\hat{D}(-\boldsymbol{\beta})\right] = \pi^n \delta(\boldsymbol{\alpha} - \boldsymbol{\beta}). \tag{3.38}$$

Thanks to these properties, the trace of the product of two operators can be expressed as an integral in phase space of both the characteristic and

Wigner function giving the following trace rule

$$\mathrm{Tr}\left[\hat{O}_1\hat{O}_2\right] = \frac{1}{\pi^n}\int_{\mathbb{C}^n} \mathrm{d}^{2n}\boldsymbol{\alpha}\, \chi_{\hat{O}_1}(\boldsymbol{\alpha})\chi_{\hat{O}_2}(-\boldsymbol{\alpha}) \tag{3.39}$$

$$= \pi^n \int_{\mathbb{C}^n} \mathrm{d}^{2n}\boldsymbol{\alpha}\, W_{\hat{O}_1}(\boldsymbol{\alpha}) W_{\hat{O}_2}(\boldsymbol{\alpha}). \tag{3.40}$$

Let us now consider specifically the Wigner function of a density operator $W_\rho(\boldsymbol{\alpha})$. As discussed above, it can only assumes real values, and it is normalized since

$$\int_{\mathbb{C}^n} \mathrm{d}^{2n}\boldsymbol{\alpha}\, W_\rho(\boldsymbol{\alpha}) = \chi_\rho(0) = \mathrm{Tr}\left[\rho\right] = 1. \tag{3.41}$$

Contrary to a probability distribution, however, it can assume also negative values. Its interpretation as a quasi-probability distribution comes from the fact that it can be used to calculate expectation values of symmetrically ordered moments as

$$\mathrm{Tr}\left[\rho(\hat{F}(\hat{a},\hat{a}^\dagger))_S\right] = \int_{\mathbb{C}^n} \mathrm{d}\alpha\, W_\rho(\alpha) F(\alpha,\alpha^*), \tag{3.42}$$

where we use the notation $(\hat{F})_S$ to indicate the symmetric ordering of an operator $\hat{F}$, and the function F is obtained by substituting $\hat{a}$ and $\hat{a}^\dagger$ with α and α^* respectively.

The action of the creation and annihilation operator on the density matrix can be translated in phase space into the following correspondences by using simply the definition of the Wigner function and the cyclicity of the trace [32]:

$$\begin{aligned} \hat{a}\rho &\longrightarrow \left(\alpha + \frac{1}{2}\partial_{\alpha^*}\right) W_\rho(\alpha), \qquad & \hat{a}^\dagger\rho &\longrightarrow \left(\alpha^* - \frac{1}{2}\partial_\alpha\right) W_\rho(\alpha), \\ \rho\hat{a} &\longrightarrow \left(\alpha - \frac{1}{2}\partial_{\alpha^*}\right) W_\rho(\alpha), \qquad & \rho\hat{a}^\dagger &\longrightarrow \left(\alpha^* + \frac{1}{2}\partial_\alpha\right) W_\rho(\alpha). \end{aligned} \tag{3.43}$$

A bosonic system is said to be in a **Gaussian state** if its Wigner (or characteristic) function is Gaussian. As such, it is fully characterized by the first and second statistical moments of its Wigner function which is a notably simpler description compared to the infinite-dimensional density operator.

Gaussian states can be equivalently defined as the states that can be written as ground or thermal states of a stable Hamiltonian that is at most quadratic in the canonical quadratures of the system [33]. As a consequence,

Gaussian states occur whenever the physical system we are studying is either exactly harmonic or can be considered in the limit of small oscillations around some equilibrium point. Some examples are optical and optomechanical systems, trapped ions, atomic ensembles, and Bose-Einstein condensates.

The Wigner function was originally introduced as a quasi-probability distribution in a position and momentum phase space. The original definition, up to a normalization constant, coincides to what we have discussed so far when we substitute $\mathrm{Re}[\alpha] = \frac{q}{\sqrt{2}}$ and $\mathrm{Im}[\alpha] = \frac{p}{\sqrt{2}}$ and it has the notable property that its marginals give the probability distribution for quadrature measurements, for example

$$\int \mathrm{d}p\, W_\rho(q,p) = \langle q|\rho|q\rangle\,. \tag{3.44}$$

In the following we use mostly the Wigner distribution as a function of the canonical quadratures of a system and we refer to Appendix A for details of this formulation. It is possible to prove that, given a Gaussian state with first moments and covariance matrix defined in Eq. (3.23), then its characteristic and Wigner function are respectively [33, 34]

$$\chi(\boldsymbol{r}) = \mathrm{e}^{-\frac{1}{4}\boldsymbol{r}^\intercal\Omega^\intercal\sigma\Omega\boldsymbol{r}}\,\mathrm{e}^{i\boldsymbol{r}^\intercal\Omega^\intercal\langle\boldsymbol{r}\rangle}, \tag{3.45}$$

$$W(\boldsymbol{r}) = \frac{1}{\pi^n\sqrt{\mathrm{Det}\,[\sigma]}}\,\mathrm{e}^{(\boldsymbol{r}-\langle\boldsymbol{r}\rangle)^\intercal\sigma^{-1}(\boldsymbol{r}-\langle\boldsymbol{r}\rangle)}\,. \tag{3.46}$$

Since Gaussian Wigner functions are positive, one may think that Gaussian states have no quantum features and for pure states it was proven in Ref. [35] that the only pure states with positive Wigner function are Gaussian states. However, it was also shown (see for example [36–40]) that squeezed Gaussian states are an important quantum resource that can be used to improve metrology protocols, and they can also be used for entanglement generation [41, 42].

3.4 Unitary evolution of Gaussian states

The most general Hamiltonian that is linear in a system's canonical quadratures can be written, with a suitable choice of a $2n \times 1$ vector $\boldsymbol{d}$, as

$$\hat{H} = \boldsymbol{d}^\intercal\Omega\hat{\boldsymbol{r}}. \tag{3.47}$$

We know then that the evolution of the density operator is given by $\rho \mapsto \rho_t = \mathrm{e}^{-i\hat{H}t}\,\rho\,\mathrm{e}^{i\hat{H}t}$, where the unitary evolution can be written as a displacement

operator $\mathrm{e}^{-i\hat{H}t} = \hat{D}(-t\boldsymbol{d})$. The characteristic function of the evolved state can be calculated by using Eq. (A.3) and, up to a phase, it is

$$\chi_\rho(\boldsymbol{r}) \mapsto \chi_\rho(\boldsymbol{r})\, \mathrm{e}^{i\boldsymbol{r}^\intercal \Omega^\intercal (t\boldsymbol{d})}\,. \tag{3.48}$$

If we assume that the initial state is Gaussian and compare with Eq. (3.45), it becomes clear that linear Hamiltonians preserve the Gaussian character of a quantum state and act in phase space simply by a rigid translation in the form

$$\langle \boldsymbol{r} \rangle \mapsto \langle \boldsymbol{r} \rangle + \boldsymbol{d}, \tag{3.49}$$

$$\sigma \mapsto \sigma. \tag{3.50}$$

If the properties of the quantum system we are interested in depend only on the covariance matrix (such as the purity or entropy), we note that the displacement for a composite system is trivially factorizable onto its subsystems as written in Eq. (3.33) and therefore local displacement can be done without interactions between the modes or other entanglement resources that could change the global covariance matrix. It is then possible to disregard the evolution of the first moments since we can act with local displacements to set the state to have 0 as first moments [43].

A general second-order Hamiltonian can be written as

$$\hat{H} = \frac{1}{2}\hat{\boldsymbol{r}}^\intercal R \hat{\boldsymbol{r}}, \tag{3.51}$$

where R is real and symmetric matrix. We study how this bilinear Hamiltonian acts in phase space by considering again its effect on the characteristic function. It can be prove that the linear evolution induced on the quadrature operators in Heisenberg picture is given by

$$\hat{S}\hat{\boldsymbol{r}}\hat{S}^\dagger = S\hat{\boldsymbol{r}}, \tag{3.52}$$

where the operator is defined as $\hat{S} = \mathrm{e}^{i\hat{H}t} = \mathrm{e}^{\frac{i}{2}\hat{\boldsymbol{r}}^\intercal R \hat{\boldsymbol{r}}}$ (we have absorbed the time t into the definition of matrix R) and the matrix is $S = \mathrm{e}^{\Omega R}$.

As a consequence, we have the action on the displacement operator as

$$\hat{S}\hat{D}(\boldsymbol{r})\hat{S}^\dagger = \hat{D}(S^{-1}\boldsymbol{r}). \tag{3.53}$$

By studying the definition of the characteristic function we get that the Hamiltonian induces the evolution

$$\chi \mapsto \chi(S^{-1}\boldsymbol{r}). \tag{3.54}$$

It is apparent then that second-order Hamiltonians send Gaussian states into Gaussian states and the evolution is completely characterized by

$$\langle \boldsymbol{r} \rangle \mapsto S\langle \boldsymbol{r} \rangle, \tag{3.55}$$

$$\sigma \mapsto S\sigma S^{\intercal}. \tag{3.56}$$

It can be shown that all matrices S constructed in such a way belong to the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$, defined as the matrices that preserve by congruence the symplectic matrix Ω. In formula $S \in \mathrm{Sp}(2n, \mathbb{R})$ if

$$S\Omega S^{\intercal} = \Omega. \tag{3.57}$$

Symplectic transformations are also studied in connection with the phase space evolution in classical mechanics. If we consider a system with canonical coordinates and momenta $\boldsymbol{R}^{\intercal} = (q_1, p_1, \ldots, q_n, p_n)$, then the evolution of the classical system is described by Hamilton's equation

$$\dot{R}_k = \Omega_{kl} \frac{\partial H}{\partial R_l}. \tag{3.58}$$

If we perform a linear change of coordinates similar to Eq. (3.55), namely

$$\boldsymbol{R}' = S\boldsymbol{R}, \qquad S_{kl} = \frac{\partial R'_k}{\partial R_l}, \tag{3.59}$$

then the time evolution of the new coordinates is given by

$$\begin{aligned} \frac{\partial R'_k}{\partial t} &= \frac{\partial R'_k}{\partial R_l} \frac{\partial R_l}{\partial t} = S_{kl}\Omega_{lm} \frac{\partial H}{\partial R_m} \\ &= S_{kl}\Omega_{lm} \frac{\partial R'_n}{\partial R_m} \frac{\partial H}{\partial R'_n} = S_{kl}\Omega_{lm}S_{nm} \frac{\partial H}{\partial R'_n}. \end{aligned} \tag{3.60}$$

We now require that the coordinate transformation is canonical, that is $\boldsymbol{R}'$ still obeys Hamilton's equation, then the condition on matrix S is that $S_{kl}\Omega_{lm}S_{nm} = \Omega_{kn}$, which is exactly the condition that it belongs to the symplectic group.

In our case the fact that the S is symplectic also guarantees that the evolved covariance matrix is still a proper covariance matrix that satisfies Robertson-Schrödinger inequality in Eq. (3.24).

If we extend the symplectic group considering also the translation in phase space given by Eq. (3.49) then it is possible to show that all Hamiltonians of first and second order generate the whole set of symplectic transformations. Let us now give a brief overview of how such Hamiltonians are implemented in quantum optics.

The most general Hamiltonian that we are considering is

$$\hat{H} = \sum_{k=1}^{n} g_k^{(1)} \hat{a}_k^\dagger + \sum_{k>l=1}^{n} g_{kl}^{(2)} \hat{a}_k^\dagger \hat{a}_l + \sum_{k,l=1}^{n} g_{kl}^{(3)} \hat{a}_k^\dagger \hat{a}_l^\dagger + \text{h.c.} \tag{3.61}$$

The linear terms in the form $g^{(1)}\hat{a}^\dagger + \text{h.c.}$ correspond simply to unitary evolution given by displacement operators. For an optical cavity, this evolution can be obtained by driving with laser light in a strong coherent state.

The mode mixing given by terms of the form $\propto \hat{a}^\dagger \hat{b} + \text{h.c.}$ correspond to a beam splitter interaction, while the terms $\propto \hat{a}^\dagger \hat{a}$ corresponds to phase shifters that are implemented with dielectric plates.

From the third sum we have terms like $\propto g^{(3)}(\hat{a}^\dagger)^2 + \text{h.c.}$ and $\propto g^{(3)}\hat{a}^\dagger \hat{b}^\dagger +$ h.c. that correspond respectively to single-mode and two-mode squeezing. These effectively linear evolution have to be generated via non-linear media. For example the single-mode squeezing can be obtained by pumping non-linear crystals that allow for degenerate parametric down-conversion. The case of non-degenerate parametric down-conversion gives the two-mode squeezed states.

Eventually a notable characterization of Gaussian states is given by the following theorem [44, 45].

Williamson theorem Any real, symmetric and positive definite $2n \times 2n$ matrix σ can be written as

$$\sigma = SWS^\intercal, \tag{3.62}$$

where $S \in \mathrm{Sp}(2n, \mathbb{C})$ and matrix W has the form

$$W = \bigoplus_{j=1}^{n} \begin{pmatrix} d_j & 0 \\ 0 & d_j \end{pmatrix}, \tag{3.63}$$

with $d_j \in \mathbb{R}$ are the symplectic eigenvalues of σ and they are uniquely defined up to permutations. The symplectic eigenvalues can be calculated as the absolute values of the eigenvalues of $i\Omega\sigma$.

Matrix W is simply the covariance matrix of a thermal state where each mode is at temperature β_j such that $d_j = 2\bar{n}_j + 1$, $\bar{n}_j = 1/(\mathrm{e}^{\beta_j} - 1)$.

The physical interpretation of this theorem is that, up to a displacement in the first moments, any Gaussian state with covariance matrix σ can be obtained starting with a thermal state with covariance matrix W and evolving unitarily with an Hamiltonian in the form of Eq. (3.61). In practical terms, it means that in optical systems any Gaussian state is in principle reachable.

This effective characterization of Gaussian states allows to easily calculate the purity of state. In fact, by applying the trace rule we have that the purity is

$$\mu = \mathrm{Tr}\left[\rho^2\right] = \frac{1}{\sqrt{\mathrm{Det}\left[\sigma\right]}}. \tag{3.64}$$

By analyzing the spectral decomposition of a Gaussian density matrix, it is also possible to obtain the von Neumann entropy of a Gaussian state by

$$S_V(\rho) = -\mathrm{Tr}\left[\rho \log_2(\rho)\right] = \sum_{j=1}^{n} f(d_j), \tag{3.65}$$

where

$$f(x) = \frac{x+1}{2}\log_2\left(\frac{x+1}{2}\right) - \frac{x-1}{2}\log_2\left(\frac{x-1}{2}\right). \tag{3.66}$$

3.5 Gaussian measurements

A measurement is said to be Gaussian if, when applied to a Gaussian state, it produces conditional and unconditional states that are still Gaussian. We recall that the conditional state is the state obtained for a specific measurement outcome as given in Eq. (2.13), while the unconditional state is what we get if the measurement is performed, but the outcome discarded and therefore we get a statistical mixture of all the possible conditional state, as given in Eq. (2.17).

Considering the phase space representation of a Gaussian state we have that a quite natural Gaussian measurement is a quadrature measurement. The relevant observable is defined as $\hat{x}_\theta = \hat{q}\cos\theta + \hat{p}\sin\theta$ and the resolution of the identity is given by the projectors onto the eigenstates $|x_\theta\rangle$

$$\int \mathrm{d}x_\theta \, |x_\theta\rangle\langle x_\theta| = \mathbb{1}. \tag{3.67}$$

This measurement is also called homodyne due to its implementation in optics that we later discuss.

An other important example is heterodyne measurement, which is associated to the resolution given by coherent states

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha \, |\alpha\rangle\langle\alpha| = \mathbb{1}. \tag{3.68}$$

As we will see, this generalized measurement corresponds to measuring simultaneously two conjugate quadratures of a mode.

The last example we consider of Gaussian measurements is a generalized measurement scheme where a system is coupled to a meter. Consider a system that is partitioned into two subsystems of n and m modes respectively that we label A and B. The tensor product of their Hilbert spaces corresponds to direct sum in phase space. Therefore an initially Gaussian state of the composite system is characterized by the following first and second moments

$$\langle \boldsymbol{r} \rangle = \begin{pmatrix} \langle \boldsymbol{r}_A \rangle \\ \langle \boldsymbol{r}_B \rangle \end{pmatrix}, \qquad \sigma = \begin{pmatrix} \sigma_A & \sigma_{AB} \\ \sigma_{AB}^{\intercal} & \sigma_B \end{pmatrix}, \tag{3.69}$$

where σ_A is a $2n \times 2n$ matrix, σ_B is of size $2m \times 2m$. Similarly $\langle \boldsymbol{r}_A \rangle$ is $2n \times 1$, and $\langle \boldsymbol{r}_B \rangle$ is $2m \times 1$. It is possible to prove that, if we trace out mode B, then the reduced state of mode A is still in a Gaussian state characterized by $\langle \boldsymbol{r}_A \rangle$ and σ_A. In the global covariance matrix σ_{AB} represents the correlations between the two subsystems.

Suppose that B is subject to a Gaussian measurement and its conditional state after the measurement is characterized by $\langle \boldsymbol{r}_{\mathrm{m}} \rangle$ and σ_{m}. It is then possible to show [46] that A is also left in a Gaussian state and the stochastic evolution for A is given by

$$\langle \boldsymbol{r}_A \rangle \mapsto \langle \boldsymbol{r}_A \rangle - \sigma_{AB}(\sigma_B + \sigma_{\mathrm{m}})^{-1} \left(\langle \boldsymbol{r}_B \rangle - \langle \boldsymbol{r}_{\mathrm{m}} \rangle \right), \tag{3.70}$$

$$\sigma_A \mapsto \sigma_A - \sigma_{AB}(\sigma_B + \sigma_{\mathrm{m}})^{-1} \sigma_{AB}^{\intercal}. \tag{3.71}$$

We have that the evolution of the first moments of A depends on the random measurement outcome $\langle \boldsymbol{r}_{\mathrm{m}} \rangle$, while the evolution of its covariance matrix is deterministic. This is a feature that is common in all stochastic evolutions of Gaussian states induced by measurement.

As a specific example of generalized measurement we consider a projective position measurement onto subsystem B which consists of $m = 1$ mode. This measurement can be thought of as the projection onto a Gaussian state that is infinitely narrow in $\hat{q}$ and has correspondingly large uncertainty in $\hat{p}$ in accordance with Heisenberg principle. We can then apply Eqs. (3.70-3.71) with σ_{m} in this limit to obtain, in accordance with [46, 47], that the n mode A system evolves as

$$\langle \boldsymbol{r}_A \rangle \mapsto \langle \boldsymbol{r}_A \rangle - \sigma_{AB}(\pi \sigma_B \pi)^{\mathrm{MP}} \left(\langle \boldsymbol{r}_B \rangle - \langle \boldsymbol{r}_{\mathrm{m}} \rangle \right), \tag{3.72}$$

$$\sigma_A \mapsto \sigma_A - \sigma_{AB}(\pi \sigma_B \pi)^{\mathrm{MP}} \sigma_{AB}^{\intercal}, \tag{3.73}$$

where $\pi = \mathrm{diag}(1, 0)$ and $(\cdot)^{\mathrm{MP}}$ indicates the Moore-Penrose pseudo-inverse, so that we have $(\pi \sigma_B \pi)^{\mathrm{MP}} = \mathrm{diag}([\sigma_B]_{11}^{-1}, 0)$.

If on the other hand we consider a heterodyne measurement, then for an outcome α we have that the conditional state of B is characterized by $\langle \boldsymbol{r}_{\mathrm{m}} \rangle = \sqrt{2}(\mathrm{Re}[\alpha], \mathrm{Im}[\alpha])^{\intercal}$ and $\sigma_{\mathrm{m}} = \mathbb{1}$.

The implementation of these two types of measurement will be discussed in more details in Section 4.2, where they will be also considered in the more general context of times-continuous monitoring discussed in Chapter 5

Chapter 4

Master equations and quantum trajectories

In this Chapter we present the description of an open quantum system interacting with a Markovian environment in the form of a master equation. We then discuss how monitoring a system induces an evolution that is given by a stochastic master equation. Eventually, we focus on the description of relevant stochastic master equations for quantum optics.

4.1 Lindblad master equation

The evolution of an open quantum system can be described by considering the global unitary evolution of the system interacting with an environment, and then by tracing out the environment's degrees of freedom. In general this problem is quite complex, but, under some assumptions that are often realistically realized, the evolution of the system retains its linearity and assumes the so called Lindblad form.

We assume the weak coupling regime, also called Born approximation, which means that the interaction of system and environment can be treated perturbatively. Furthermore Markov approximation is assumed, meaning that the environment has a much larger number of degrees of freedom compared to the system of interest, and its evolution is also much faster, so that the environment can be considered memoryless. More precisely, this means that correlations between the system and environment on the time scale of the system's evolution can be neglected. Under these conditions

it was proven that the evolution is always given by the Lindblad master equation [48]

$$\mathrm{d}\rho = -i\big[\hat{H},\rho\big]\,\mathrm{d}t + \sum_h \mathcal{D}[\hat{c}_h]\rho\,\mathrm{d}t\,, \tag{4.1}$$

where $\hat{H}$ is the Hamiltonian of the system, the operators $\hat{\boldsymbol{c}}^\intercal = (\hat{c}_1,\ldots,\hat{c}_m)$ are the Lindblad operators that couple the system with the environment modes, and the dissipation superoperator is defined as

$$\mathcal{D}[\hat{c}]\rho = \hat{c}\rho\hat{c}^\dagger - \frac{1}{2}\Big\{\hat{c}^\dagger\hat{c},\rho\Big\}. \tag{4.2}$$

This evolution described by Eq. (4.1) is invariant for unitary transformation of the Lindblad operators [49] and for shift by a number. That is, given a unitary matrix T and complex numbers χ_k, we obtain the same evolution under the transformations

$$\hat{\boldsymbol{c}} \mapsto T\hat{\boldsymbol{c}}, \tag{4.3}$$

$$\hat{c}_k \mapsto \hat{c}_k + \chi_k, \qquad \hat{H} \mapsto \hat{H} - \frac{i}{2}\left(\chi_k^*\hat{c}_k - \chi_k\hat{c}_k^\dagger\right). \tag{4.4}$$

Equation (4.1) describes a CPTP map for the system, since it satisfies all the requirements discussed in Section 2.3, therefore it can also be represented in Kraus form by the Kraus operators

$$\hat{K}_0 = 1 - i\hat{H}\,\mathrm{d}t - \sum_h \frac{1}{2}\hat{c}_m^\dagger\hat{c}_m\,\mathrm{d}t\,, \qquad \hat{K}_m = \hat{c}_m\sqrt{\mathrm{d}t}. \tag{4.5}$$

When studying a Markovian master equation, instead of tracing out and ignoring the environment, it is often possible and desirable to measure its degrees of freedom in order to perform effectively a generalized measurement on the system. This probing induces a stochastic evolution of the system, since the result of the measurement at every time is random. In the case of continuous measurements a specific sequence of random outcomes is called a *measurement record* and it corresponds to a specific realization of the system's dynamics that is called a *quantum trajectory*.

The stochastic evolution is given by the following stochastic master equation [49]

$$\mathrm{d}\rho = -i\big[\hat{H},\rho\big]\,\mathrm{d}t + \sum_h \mathcal{D}[\hat{c}_h]\,\mathrm{d}t + \sum_h \sqrt{\eta_h}\mathcal{H}[\hat{c}_h]\rho\,\mathrm{d}W_h(t)\,, \tag{4.6}$$

where the measurement superoperator is

$$\mathcal{H}[\hat{c}]\rho = \hat{c}\rho + \rho\hat{c}^\dagger - \mathrm{Tr}\left[\rho(\hat{c}+\hat{c}^\dagger)\right]\rho, \tag{4.7}$$

$0 \leq \eta_h \leq 1$ are the efficiency with which each dissipation channel is measured.

$\mathrm{d}W_h(t)$ are stochastic Wiener increments that obey the rules of Itô calculus. They are normally distributed with mean 0 and variance $\mathrm{d}t$ and uncorrelated from each other. If we indicate with E the expectation value, we have

$$\mathrm{E}\left[\mathrm{d}W_h(t)\right] = 0, \qquad \mathrm{E}\left[\mathrm{d}W_h(t)\,\mathrm{d}W_k(t)\right] = \delta_{hk}\,\mathrm{d}t\,, \tag{4.8}$$

$$(\mathrm{d}W(t))^2 = \mathrm{d}t\,. \tag{4.9}$$

In particular, Eq. (4.9) expresses the notable property that, while $\mathrm{d}W_h(t)$ is a random variable, its square is not and has a determined value. As an important consequence that we will later apply, when studying stochastic differential equations that contain Wiener increments we need to expand all terms to second order to retain all terms of first order in $\mathrm{d}t$.

4.2 Measurements in quantum optics

While in the previous part we gave a very general and abstract evolution for a system that is continuously probed, we would like in this Section to describe in more detail how some measurements are implemented in optical settings.

Photodetection A photon can be experimentally detected due to its interaction with atoms or molecules. In photodetectors the ionization of an atom or the promotion of an electron to the conduction band in a solid caused by a photon starts an avalanche process that creates enough electric current to be detected.

Even though photodetection is not a Gaussian measurement, it is nevertheless a fundamental tool in quantum optics, and it is necessary for other measurement schemes such as photon counting and the practical implementation of quadrature measurement, as we will see in the next Section.

Let us consider a typical example of a Markovian system, that of an optical cavity with damping rate γ that is leaking into the environment.

The unitary global interaction between the cavity and the environment in an infinitesimal $\mathrm{d}t$ is given by (see Appendix B, [50])

$$\hat{U}(\mathrm{d}t) = \exp\left[-i\hat{H}\,\mathrm{d}t + (\hat{b}^\dagger\hat{c} - \hat{c}^\dagger\hat{b})\sqrt{\mathrm{d}t}\right]. \tag{4.10}$$

where $\hat{a}$ is the annihilation operator of the optical mode, $\hat{c} = \sqrt{\gamma}\hat{a}$, $\hat{b}$ is the environment mode and we have the free Hamiltonian of the system $\hat{H} = \omega\hat{a}^\dagger\hat{a}$.

We can perform photodetection on the light that is leaking out of the cavity. Since the interval $\mathrm{d}t$ is infinitesimally small, in each interval the outcome of the photodetection will be either 0 or 1 photon. According to the theory of generalized measurements (see Eq. (2.11)) the measurement operators for the system expanded to first order in $\mathrm{d}t$ are

$$\hat{M}_1 = \langle 1|\hat{U}(\mathrm{d}t)|0\rangle = \hat{c}\sqrt{\mathrm{d}t}, \tag{4.11}$$

$$\hat{M}_0 = \langle 0|\hat{U}(\mathrm{d}t)|0\rangle = 1 - \left(i\hat{H} + \frac{1}{2}\hat{c}^\dagger\hat{c}\right)\mathrm{d}t\,. \tag{4.12}$$

If we introduce the stochastic variable $\mathrm{d}N(t)$ that is the measurement outcome at t then it must satisfy

$$\mathrm{d}N(t)^2 = \mathrm{d}N\,(t), \tag{4.13}$$

$$\mathrm{E}\left[\mathrm{d}N(t)\right] = \mathrm{Tr}\left[\rho(t)\hat{M}_1^\dagger\hat{M}_1\right] = \mathrm{Tr}\left[\rho\hat{c}^\dagger\hat{c}\right]\mathrm{d}t\,. \tag{4.14}$$

The unnormalized density matrix that is conditioned on a measurement can be written as

$$\tilde{\rho}(t+\mathrm{d}t) = \mathrm{d}N(t)\,\hat{M}_1\rho\hat{M}_1^\dagger + (1-\mathrm{d}N(t))\hat{M}_0\rho\hat{M}_0^\dagger \tag{4.15}$$

We can normalize the state to obtain the stochastic master equation

$$\mathrm{d}\rho = -i\big[\hat{H},\rho\big]\,\mathrm{d}t - \frac{1}{2}\mathcal{H}[\hat{c}^\dagger\hat{c}]\rho\,\mathrm{d}t + \mathcal{G}[\hat{c}]\rho\,\mathrm{d}N(t)\,, \tag{4.16}$$

where the superoperator $\mathcal{G}$ is defined as

$$\mathcal{G}[\hat{c}]\rho = \frac{\hat{c}\rho\hat{c}^\dagger}{\mathrm{Tr}\left[\hat{c}\rho\hat{c}^\dagger\right]} - \rho. \tag{4.17}$$

Homodyne detection In a homodyne detection scheme the signal $\hat{c}$ is mixed through a beam splitter with a local oscillator in a strong coherent state $|\alpha\rangle$ with the same frequency. As shown in Fig. 4.1, at the output of the beam splitter the intensities are measured and subtracted from each other. The resulting homodyne photocurrent is given by

$$\hat{I} = \hat{d}^\dagger\hat{d} - \hat{e}^\dagger\hat{e} = \hat{c}^\dagger\hat{b} + \hat{c}\hat{b}^\dagger. \tag{4.18}$$

Since $\hat{b}$ is in a strong coherent state with $\alpha = |\alpha|\,\mathrm{e}^{i\varphi}$, we can approximate the observable that is being measured with

$$\langle\alpha|\hat{c}^\dagger\hat{b} + \hat{c}\hat{b}^\dagger|\alpha\rangle \approx |\alpha|\sqrt{2}\frac{\hat{c}\,\mathrm{e}^{-i\varphi} + \hat{c}^\dagger\,\mathrm{e}^{i\varphi}}{\sqrt{2}} = |\alpha|\sqrt{2}\hat{x}_\varphi. \tag{4.19}$$

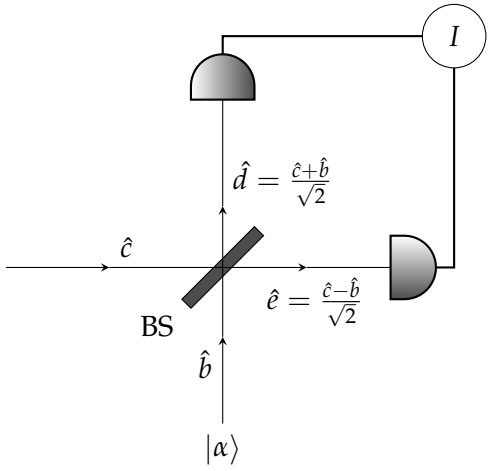

Figure 4.1 – Scheme for homodyne detection: the signal and the local oscillator in a strong coherent state $|\alpha\rangle$ are mixed with a 50:50 beam splitter. At the outputs the intensity is measured and the relevant signal is the difference between the intensities.

Therefore, up to rescaling by a constant, the measured current corresponds to measuring quadrature $\hat{x}_\varphi$ of the signal.

A more detailed analysis of homodyne detection, including the possibility of imperfect detection efficiency, is given in Refs. [51, 52] and gives the master equation

$$\mathrm{d}\rho = -i\big[\hat{H},\rho\big] + \mathcal{D}[\hat{c}]\rho\,\mathrm{d}t + \sqrt{\eta}\,\mathcal{H}[\hat{c}]\rho\,\mathrm{d}W(t)\,, \tag{4.20}$$

where $0 \leq \eta \leq 1$ is the measurement efficiency, the Lindblad operator is $\hat{c} = \sqrt{\gamma}\hat{a}$ with γ the damping constant and $\mathrm{d}W(t)$ is a Wiener increment.

In the case of unit efficiency one can find the measurement operator from master equation (4.20). The possible outcomes of the measurement in an interval $\mathrm{d}t$ is the measurement current $\mathrm{d}y$ that can be written as

$$\mathrm{d}y(t) = \mathrm{Tr}\left[\rho(\hat{c}+\hat{c}^\dagger)\right] + \mathrm{d}W(t)\,, \tag{4.21}$$

which can be interpreted as an average value determined by the current state of the system (consistently with the Markovian hypothesis) and a random fluctuation with the statistics of a Wiener process. The corresponding

measurement operators in terms of the Lindblad operator are therefore

$$\hat{M}_{\mathrm{d}y} = \frac{1}{\sqrt[4]{2\pi\,\mathrm{d}t}} \exp\left[-\frac{\mathrm{d}y^2}{4\,\mathrm{d}t}\right]\left(1 - i\hat{H}\,\mathrm{d}t - \frac{1}{2}\hat{c}^\dagger\hat{c}\,\mathrm{d}t + \hat{c}\,\mathrm{d}y\right), \tag{4.22}$$

and the are correctly normalized to have $\int \mathrm{d}(\mathrm{d}y)\, \hat{M}^\dagger_{\mathrm{d}y}\hat{M}_{\mathrm{d}y} = \mathbb{1}$ to first order in $\mathrm{d}t$. The conditional evolution for ρ is then simply given by

$$\rho(t + \mathrm{d}t) = \frac{\hat{M}_{\mathrm{d}y}\rho\hat{M}^\dagger_{\mathrm{d}y}}{\mathrm{Tr}\left[\hat{M}_{\mathrm{d}y}\rho\hat{M}^\dagger_{\mathrm{d}y}\right]}. \tag{4.23}$$

In the case of inefficient measurement, we can consider the model shown in Fig. 4.2. The signal first goes through a beam splitter of transmissivity η that has the vacuum state in the other input, so that a fraction $1 - \eta$ of the signal is lost. We then perform an ideal homodyne detection on the remaining signal. The measurement current is

$$\mathrm{d}y(t) = \sqrt{\eta}\,\mathrm{Tr}\left[\rho(\hat{c} + \hat{c}^\dagger)\right] + \mathrm{d}W(t)\,, \tag{4.24}$$

and the normalized measurement operators are [53, 54]

$$\hat{M}_{\mathrm{d}y} = \sqrt[4]{\frac{\eta}{2\pi\,\mathrm{d}t}} \exp\left[-\frac{\eta\,\mathrm{d}y^2}{4\,\mathrm{d}t}\right]\left(1 - i\hat{H}\,\mathrm{d}t - \frac{1}{2}\hat{c}^\dagger\hat{c}\,\mathrm{d}t + \sqrt{\eta}\hat{c}\,\mathrm{d}y\right). \tag{4.25}$$

In order to have the complete master equation for homodyne measurement give in Eq. (4.20) the normalized evolution in terms of Kraus operators is given by

$$\rho(t + \mathrm{d}t) = \frac{\hat{M}_{\mathrm{d}y}\rho\hat{M}^\dagger_{\mathrm{d}y}}{\mathrm{Tr}\left[\hat{M}_{\mathrm{d}y}\rho\hat{M}^\dagger_{\mathrm{d}y}\right]} + (1 - \eta)\left(\hat{c}\rho\hat{c}^\dagger - \mathrm{Tr}\left[\hat{c}\rho\hat{c}^\dagger\right]\rho\right)\mathrm{d}t\,, \tag{4.26}$$

where the first part is the measurement back action, while the last term accounts for the fraction of the signal that is lost and not detected.

Heterodyne detection In the heterodyne detection scheme the setup is similar to the one for homodyne detection, but the local oscillator is chosen to be of a frequency ω_{LO} different than the signal's frequency ω. This effectively corresponds to substituting in Eq. (4.19) $\hat{c} \to \mathrm{e}^{-i\Delta t}\,\hat{c}$, where $\Delta = \omega - \omega_{\mathrm{LO}}$. So the stochastic master equation (4.20) becomes

$$\mathrm{d}\rho = -i\left[\hat{H}, \rho\right] + \mathcal{D}[\hat{c}]\rho\,\mathrm{d}t + \sqrt{\eta}\,\mathcal{H}[\mathrm{e}^{-i\Delta t}\,\hat{c}]\rho\,\mathrm{d}W(t)\,, \tag{4.27}$$

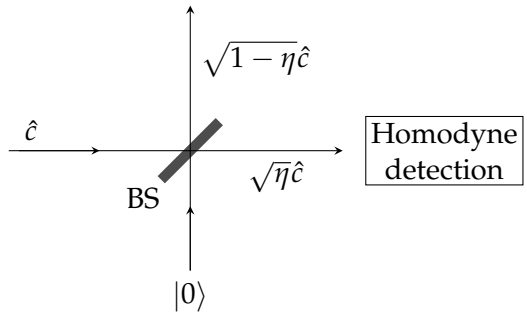

Figure 4.2 – Model for an inefficient homodyne detection. The signal goes through a beam splitter $\eta : 1 - \eta$ and only one output is sent to the (ideal) measurement apparatus.

where we note that the dissipation superoperator is invariant if we multiply $\hat{c}$ by a phase, that is $\mathcal{D}[\mathrm{e}^{-i\varphi}\,\hat{c}] = \mathcal{D}[\hat{c}]$.

It is shown in Ref. [51] that if we assume that the detuning $\Delta = \omega - \omega_{\mathrm{LO}}$ is much larger than the typical time scale of evolution of the system, we can take an average over times $\delta t \approx \Delta^{-1}$ where the system evolves very slowly. It is then useful to define the Fourier cosine and sine components of the Wiener noise that comes from the measurement as

$$\mathrm{d}W_c + i\,\mathrm{d}W_s = \sqrt{2}\int_t^{t+\delta t} \mathrm{d}W(s)\,\mathrm{e}^{i\Delta s}\,. \tag{4.28}$$

With this definition we can prove that for time scales of at least δt the stochastic variables $\mathrm{d}W_c$ and $\mathrm{d}W_s$ are Wiener increments with variance δt and that they are uncorrelated. The master equation can be rewritten as

$$\mathrm{d}\rho = -i\big[\hat{H},\rho\big] + \mathcal{D}[\hat{c}]\rho\,\mathrm{d}t + \sqrt{\frac{\eta}{2}}\mathcal{H}[\hat{c}]\rho\,\mathrm{d}W_c + \sqrt{\frac{\eta}{2}}\mathcal{H}[-i\hat{c}]\rho\,\mathrm{d}W_s\,. \tag{4.29}$$

This gives an other interpretation of heterodyne measurement. We have in fact the same master equation if we first split in half the signal with a system similar to the one in Fig. 4.2 with transmissivity $\eta = 1/2$, and we then perform homodyne measurement on conjugate quadratures for the two outputs.

Chapter 5

Gaussian forward and backward evolution

In this Chapter we discuss the past quantum state theory of a monitored Markovian system and specialize it to the case where a Gaussian description for both the density matrix ρ and the effect matrix E, introduced in Section 2.4, is available. The content of this Chapter is based on and partially overlaps with [1]. The Chapter ends with some examples of applications for the theory that is developed.

In order to completely determine the evolution of a Gaussian state we need only to specify the dynamics of its first and second statistical moments and in the following we choose to describe the phase space with real variables, as previously mentioned, and we refer to Appendix A for the specific detail of the notation. This will allow us to give an efficient description for a past quantum state by means of the previously introduced first moments $\langle \boldsymbol{r} \rangle$ and covariance matrix σ for ρ and analogously defined first and second moments for E.

Let us consider the stochastic Lindblad master equation for a system with n bosonic modes

$$\mathrm{d}\rho = -i\big[\hat{H}, \rho\big]\,\mathrm{d}t + \sum_h \mathcal{D}[\hat{c}_h]\,\mathrm{d}t + \sum_h \sqrt{\eta_h}\mathcal{H}[\hat{c}_h]\rho\,\mathrm{d}W_h(t) \tag{5.1}$$

where we assume a purely quadratic Hamiltonian $\hat{H} = \frac{1}{2}\hat{\boldsymbol{r}}^\intercal R \hat{\boldsymbol{r}}$, that the m dissipation channels are linear in the system's canonical quadratures and can therefore be written as $\hat{\boldsymbol{c}} = \widetilde{C}\hat{\boldsymbol{r}}$, where we have defined $\hat{\boldsymbol{c}}^\intercal = (\hat{c}_1, \ldots, \hat{c}_m)$ and $\widetilde{C}$ is a complex $m \times 2n$ matrix. As discussed in the previous Chapter each

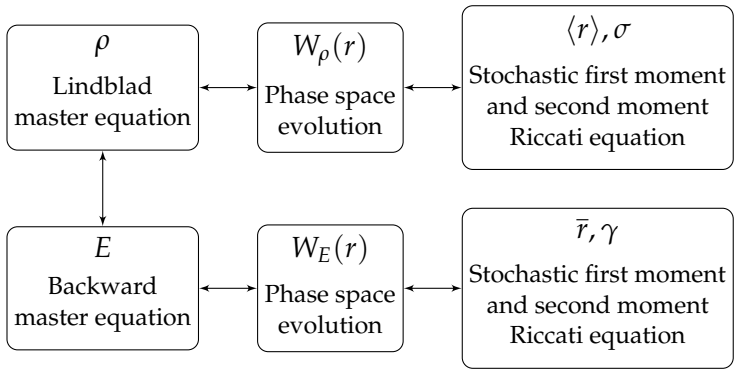

Figure 5.1 – The phase space formalism allows for a Gaussian density matrix to be fully characterized by the first and second moments of its Wigner function. The conditional evolution for ρ induces a corresponding backward evolution for the so-called effect matrix E according to the theory presented in Section 2.4. Under the same assumptions the Wigner function associated with the effect matrix E is also Gaussian and we have therefore a complete description of a Gaussian past quantum state.

dissipation channel $\hat{c}_h$ can be monitored with some efficiency $0 \leq \eta_h \leq 1$, with $\eta_h = 0$ describing the case of no measurement on the corresponding mode. The stochastic quantities $\mathrm{d}W_h(t)$ are related to the measured current by

$$\mathrm{d}W_h(t) = \mathrm{d}y_h(t) - \sqrt{\eta_h}\,\mathrm{Tr}\left[\rho(\hat{c}_h + \hat{c}_h^\dagger)\right]\mathrm{d}t\,. \tag{5.2}$$

It is convenient to define the matrix of efficiencies as $\boldsymbol{\eta} = \mathrm{diag}(\eta_1, \ldots, \eta_m)$ and the vector of stochastic quantities $\mathrm{d}\boldsymbol{W}(t)^\intercal = (\mathrm{d}W_1(t), \ldots, \mathrm{d}W_m(t))$.

A was shown in Ref. [5], the evolution for the effect matrix $E(t)$ corresponding to this master equation is given by

$$\begin{aligned}\mathrm{d}\tilde{E}(t) =& i\left[\hat{H}, E(t)\right] + \sum_h \mathcal{D}^\dagger[\hat{c}_h]E\,\mathrm{d}t + \\ &+ \sum_h \sqrt{\eta_h}\left(\hat{c}_h^\dagger E(t) + E(t)\hat{c}_h\right)\mathrm{d}y_h(t - \mathrm{d}t)\,,\end{aligned} \tag{5.3}$$

where we define $\mathrm{d}\tilde{E}(t) = \tilde{E}(t - \mathrm{d}t) - E(t)$, the tilde indicates the fact that this master equation is not trace-preserving and the adjoint dissipation su-

peroperator is defined as

$$\mathcal{D}^\dagger[\hat{c}]E = \hat{c}^\dagger E\hat{c} - \frac{1}{2}\Big\{\hat{c}^\dagger\hat{c}, E\Big\}. \tag{5.4}$$

When studying the past quantum state numerically the normalization of the effect matrix can be neglected since the past probability distribution in Eq. (2.26) contains explicitly a normalization constant. However, since we are interested in formulating a Gaussian description for the effect matrix, in order to have the correct Wigner distribution we need first of all to normalize the evolution given by Eq. (5.3) with the subtlety that Itô calculus is needed to perform the correct expansion.

The resulting trace-preserving backward master equation is

$$\begin{aligned} \mathrm{d}E = i\big[\hat{H}, E\big]\,\mathrm{d}t + \sum_h \Big(\mathcal{D}^\dagger[\hat{c}_h]E - \Big(\overline{c_h c_h^\dagger} - \overline{c_h^\dagger c_h}\Big)\,E\Big)\,\mathrm{d}t + \\ + \sum_h \sqrt{\eta_h}\,\mathcal{H}[\hat{c}_h^\dagger]E\,\mathrm{d}s_h(t-\mathrm{d}t)\,, \end{aligned} \tag{5.5}$$

where we define the notation for $\bar{c}$ and the stochastic variable $\mathrm{d}s$ respectively as

$$\bar{c} = \mathrm{Tr}\,[\hat{c}E]\,, \qquad \mathrm{d}s_h\,(t) = \mathrm{d}y_h(t) - \sqrt{\eta_h}\,\mathrm{Tr}\left[E(\hat{c}_h^\dagger + \hat{c}_h)\right]\mathrm{d}t\,. \tag{5.6}$$

Similarly to what we do for the forward equation let us define the vector of stochastic quantities $\mathrm{d}\boldsymbol{s}^\mathsf{T} = (\mathrm{d}s_1\,, \ldots, \mathrm{d}s_m)$.

In Section 3.4 we have already shown that the condition for a unitary evolution to preserve Gaussianity is that the Hamiltonian is at most bilinear in the system quadrature operators and this is true for both the forward and backward evolution. We will in the following discuss the effect of dissipation and measurement in phase space and then derive the evolution of the system in the Gaussian representation.

When translating the master equation into phase space description, it turns out that the fact that $\hat{\boldsymbol{c}}$ is linear in $\hat{\boldsymbol{r}}$ is fundamental to ensure that an initially Gaussian state stays Gaussian. To clarify this point let us consider for simplicity a single bosonic mode with annihilation operator $\hat{a}$ subject only to dissipation. The generalization is straightforward.

$$\dot{\rho} = \gamma\mathcal{D}[\hat{a}]\rho = \gamma\left(\hat{a}\rho\hat{a}^\dagger - \frac{1}{2}\Big\{\hat{a}^\dagger\hat{a}, \rho\Big\}\right). \tag{5.7}$$

The effect that this evolution has in the phase space can easily be calculated using Eq. (3.43) and it reads

$$\frac{\mathrm{d}}{\mathrm{d}t}W_\rho(\alpha, \alpha^*, t) = \frac{\gamma}{2}\left(\partial_\alpha\alpha + \partial_{\alpha^*}\alpha^* + \partial^2_{\alpha\alpha^*}\right)W_\rho(\alpha, \alpha^*, t). \tag{5.8}$$

This can be recognized as an example of a Fokker-Planck equation for the Wigner distribution (see for example appendix B.5 in Ref. [51]). Specifically this corresponds to the case of linear drift and constant diffusion, in which case the equation is also called Ornstein-Uhlenbeck equation and is known to admit solutions that remain Gaussian during the evolution.

In the case of the effect matrix it is important to note that only the trace-preserving form of the master equation given in Eq. (5.5) leads to a similar Ornstein-Uhlenbeck equation for the Wigner function or E. One obtains that the backward evolution of $W_E(\alpha, \alpha^*, t)$ has opposite drift coefficient and same diffusion. In particular this means that the Lindbladian preserves the Gaussianity for both the forward and backward evolution.

As for the effect of the measurement superoperators, it can be show that, for linear dissipation operators $\hat{\boldsymbol{c}}$, the monitoring of the environment can be described with the theory of Gaussian measurements presented in Section 3.5, where subsystem A is the system of interest and subsystem B is the environment mode coupled to it. Therefore we have that also the measurement terms in master equations (5.1) and (5.5) preserve Gaussianity.

One can translate the forward and backward stochastic master equation into evolutions of first and second moments of the corresponding Wigner distributions, as represented in the scheme in Fig. 5.1. The derivation of such equations can be found in Appendix C. The result for the evolution of the density matrix's $(\langle \boldsymbol{r} \rangle, \sigma)$, and for the effect matrix's $(\bar{\boldsymbol{r}}, \gamma)$ is given by[1]:

$$\mathrm{d}\langle \boldsymbol{r} \rangle = A\langle \boldsymbol{r}(t) \rangle + (\sigma(t)\,\mathrm{Re}[\widetilde{C}]^\intercal - \Omega\,\mathrm{Im}[\widetilde{C}]^\intercal)\sqrt{\boldsymbol{\eta}}\,\mathrm{d}\boldsymbol{W}(t)\,, \tag{5.9}$$

$$\mathrm{d}\bar{\boldsymbol{r}} = -A\overline{\boldsymbol{r}(t)}\,\mathrm{d}t + (\gamma(t)\,\mathrm{Re}[\widetilde{C}]^\intercal + \Omega\,\mathrm{Im}[\widetilde{C}]^\intercal)\sqrt{\boldsymbol{\eta}}\,\mathrm{d}\mathbf{s}(t - \mathrm{d}t)\,, \tag{5.10}$$

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}t} =& A\sigma + \sigma A^\intercal + D \\ &- 2(\sigma\,\mathrm{Re}[\widetilde{C}]^\intercal - \Omega\,\mathrm{Im}[\widetilde{C}]^\intercal)\boldsymbol{\eta}(\sigma\,\mathrm{Re}[\widetilde{C}]^\intercal - \Omega\,\mathrm{Im}[\widetilde{C}]^\intercal)^\intercal,\end{aligned} \tag{5.11}$$

$$\begin{aligned}\frac{\mathrm{d}\gamma}{\mathrm{d}t} =& -A\gamma - \gamma A^\intercal + D \\ &- 2(\gamma\,\mathrm{Re}[\widetilde{C}]^\intercal + \Omega\,\mathrm{Im}[\widetilde{C}]^\intercal)\boldsymbol{\eta}(\gamma\,\mathrm{Re}[\widetilde{C}]^\intercal + \Omega\,\mathrm{Im}[\widetilde{C}]^\intercal)^\intercal,\end{aligned} \tag{5.12}$$

where the definition of the drift matrix is $A = \Omega\left(R + \mathrm{Im}[\widetilde{C}^\dagger\widetilde{C}]\right)$ and the diffusion matrix is $D = -2\Omega\,\mathrm{Re}[\widetilde{C}^\dagger\widetilde{C}]\Omega$.

[1] Note that in the notation of [1] there is a mistake and to have the correct evolutions of the moments one needs to write $N = \mathrm{Im}[\widetilde{C}]\Omega^\intercal$.

The same type of equations has been studied in the context of classical Gaussian random processes [55], and the corresponding theory is called Kalman filtering. In particular the non-linear matrix equations are in a form that is called [51] *differential Riccati equation* and is studied in the theory of classical control.

A notable feature of the dynamics of monitored Gaussian states, as discussed in Section 3.5, is that the evolution of the first moments is stochastic and depends on the random measurement outcome, while the evolution of the covariance matrix is deterministic. Furthermore, for both the forward and backward evolution the effect of measurement is to reduce the uncertainty on the system, since the second lines of Eq. (5.11) and Eq. (5.12) are positive definite.

As anticipated in the discussion about the phase space representation of past quantum states the evolution of the effect matrix compared to the density matrix has opposite drift and same diffusion.

It is worth recalling that despite the parallels between ρ and E, the meaning of the effect matrix is given by Eq. (2.26): it represents a Bayesian update of the prior knowledge given by ρ and, unless $\rho = \mathbb{1}$, it can not be used to calculate probability of measurement outcomes by itself.

5.1 Retrodiction of quadratures

The past probability distribution in the case of retrodiction of a projective measurement $\hat{M}_n = |n\rangle\langle n|$ reduces to

$$\mathrm{Pr}_{\mathrm{p}}(n) \propto \langle n|\rho|n\rangle \ \langle n|E|n\rangle \,, \tag{5.13}$$

so that the past probability distribution can be evaluated by calculating the matrix element in the observable's eigenbasis for Gaussian ρ and E in a similar way. For example the Fock-state content of squeezed and displaced states has been studied extensively in the literature (see e.g. [56]) and analytical results are provided.

As discussed in Section 3.5, quadrature measurements are particularly easy to treat in Gaussian formalism. We consider a single mode for simplicity. Let the density operator and the effect matrix be respectively characterized by the first moments $\langle \boldsymbol{r}\rangle^{\mathsf{T}} = (\langle q\rangle, \langle p\rangle)$ and $\bar{\boldsymbol{r}} = (\overline{q}, \overline{p})^{\mathsf{T}}$, and covariance matrices

$$\sigma = \begin{pmatrix} \sigma_{qq} & \sigma_{qp} \\ \sigma_{qp} & \sigma_{pp} \end{pmatrix}, \qquad \text{and} \qquad \gamma = \begin{pmatrix} \gamma_{qq} & \gamma_{qp} \\ \gamma_{qp} & \gamma_{pp} \end{pmatrix}. \tag{5.14}$$

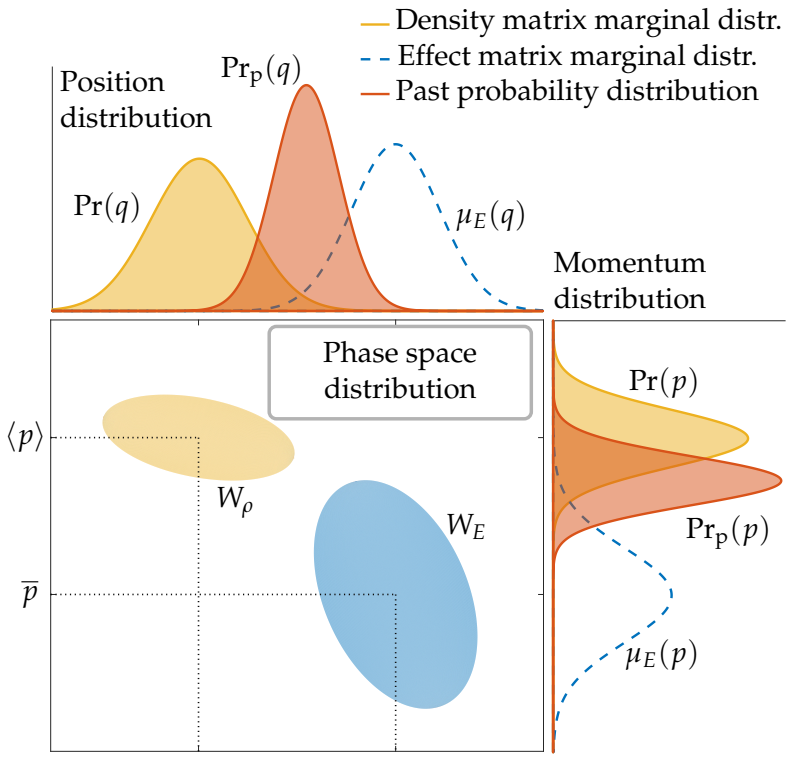

Figure 5.2 – A Gaussian past quantum state is represented in phase space by the Wigner functions for the density matrix ρ and the effect matrix E. Their covariance matrix ellipses project onto Gaussian marginal distributions of the separate quadratures and the marginals of W_ρ give exactly the probability distribution for quadrature measurements. The product of the marginal distributions yields the retrodicted probability distributions which are also Gaussian functions with the mean and variance given in Eq. (5.18).

Then the past probability distribution for a projective quadrature measurement is given by

$$\mathrm{Pr}_{\mathrm{p}}(x_\theta) = \frac{\langle x_\theta|\rho|x_\theta\rangle \ \langle x_\theta|E|x_\theta\rangle}{\int \mathrm{d}x'_\theta \ \langle x'_\theta|\rho|x'_\theta\rangle \ \langle x'_\theta|E|x'_\theta\rangle}, \tag{5.15}$$

and can be easily calculated using the marginals of the Wigner function. The resulting distribution is a Gaussian with mean $x_{\theta,\mathrm{p}}$ and variance $\mathrm{Var}(x_{\theta,\mathrm{p}}) = \Delta(x_{\theta,\mathrm{p}})/2$ given by

$$x_{\theta,\mathrm{p}} = \frac{(\langle q\rangle \cos\theta + \langle p\rangle \sin\theta)\gamma_\theta + (\overline{q}\cos\theta + \overline{p}\sin\theta)\sigma_\theta}{\gamma_\theta + \sigma_\theta}, \tag{5.16}$$

$$\frac{1}{\Delta(x_{\theta,\mathrm{p}})} = \frac{1}{\sigma_\theta} + \frac{1}{\gamma_\theta}, \tag{5.17}$$

where we have defined $\sigma_\theta = \sigma_{qq}\cos^2\theta - 2\sigma_{qp}\sin\theta\cos\theta + \sigma_{pp}\sin^2\theta$ and analogously for γ_θ. It is then clear that the inclusion of future monitoring represented by the effect matrix can only reduce the variance on any quadrature measurement as is represented in Fig. 5.2 for position and momentum quadratures.

When measuring the position quadrature the formulae reduce to

$$q_{\mathrm{p}} = \frac{\langle q\rangle\gamma_{qq} + \overline{q}\sigma_{qq}}{\sigma_{qq} + \gamma_{qq}}, \qquad \frac{1}{\Delta(q_{\mathrm{p}})} = \frac{1}{\sigma_{qq}} + \frac{1}{\gamma_{qq}}. \tag{5.18}$$

5.2 Decaying oscillator subject to homodyne detection

Let us now apply the past quantum state theory to study the dynamics of a leaky cavity subject to homodyne detection. Let the free Hamiltonian of the system be $\hat{H} = \Omega\hat{a}^\dagger\hat{a}$ and Γ be the decay rate into a zero-temperature bath. We assume that the initial state is a displaced thermal state.

The stochastic master equation for the system reads

$$\mathrm{d}\rho = -i\Omega\left[\hat{a}^\dagger\hat{a},\rho\right] + \Gamma\mathcal{D}[\hat{a}]\rho\,\mathrm{d}t + \sqrt{\Gamma\eta}\,\mathcal{H}[\hat{a}]\rho\,\mathrm{d}W\,. \tag{5.19}$$

On can then use Equations (5.9-5.12) to have the past quantum state $(\rho(t), E(t))$ fully characterized at any time for some time interval $[0,T]$ and then use $(\langle q\rangle, \sigma_{qq})$ and $(\overline{q}, \gamma_{qq})$ to calculate a past estimation of the position of the oscillator by Eq. (5.18).

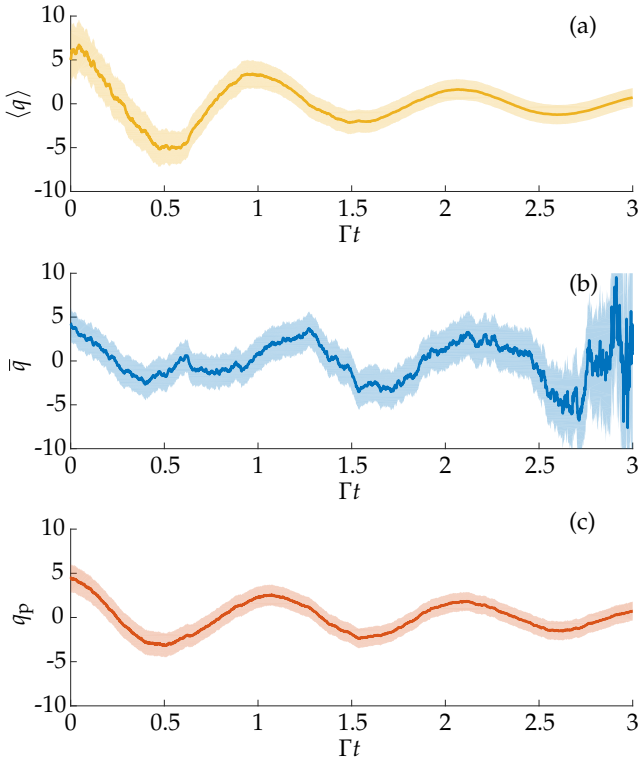

Figure 5.3 – Time evolution of: (a) forward first moment $\langle q \rangle$, (b) backward first moment $\overline{q}$, (c) past estimation of position operator q_{p}. The shaded areas in the two upper panels indicate the values given by $\sqrt{\sigma_{qq}}$ and $\sqrt{\gamma_{qq}}$ respectively, while the corresponding uncertainty on the retrodicted value is given by $(\sigma_{qq}^{-1} + \gamma_{qq}^{-1})^{-\frac{1}{2}}$ (see Eq. (5.18)). The harmonic oscillator has frequency Ω and damping constant Γ where $\Omega/\Gamma = 6$, while the efficiency of the homodyne detection is $\eta = 0.5$. The initial state is a thermal state with initial first moment $\mathbf{r}(0)^{\intercal} = (5, 0)$ and covariance matrix $\sigma(0) = 10 \times \mathbb{1}$.

The result can be seen in Fig. 5.3. We see in the top panel that the quadrature $\langle q \rangle$ has a mean value that is governed by the unitary evolution of the harmonic oscillator, the dissipation into the environment, and by the random measurement outcome from the homodyne detection. The variance σ_{qq} decreases with time (upper panel). The middle panel shows the similar stochastic and deterministic evolution of $(\overline{q}, \gamma_{qq})$ for the effect matrix, while the last panel shows our retrodicted knowledge about the position quadrature of the system. Compared to the usual forward evolution given by ρ, by including in the analysis the full past quantum state, the noise on our estimate (represented by the shaded area) is smaller at all times.

5.3 Retrodiciton beyond Heisenberg uncertainty relation

Heisenberg's uncertainty relation states a fundamental limitation to how well one can predict the outcome of the measurements of two non-commuting observables. In the case of Gaussian states it is expressed by the Robertson-Schrödinger uncertainty relation given in Eq. (3.24), which limits the precision with which we can predict the conjugate quadratures $\hat{q}$ and $\hat{p}$ at the same time. However these uncertainty relations only apply to our ability to predict future measurements and do not describe our ability to retrodict past measurements, where we have access to the system both before and after the measurement in question.

It was in fact observed in Ref. [57] that one can prepare a spin $\frac{1}{2}$ in a $\hat{\sigma}_x$ eigenstate and subsequently perform a projective measurement of $\hat{\sigma}_z$. By applying the ABL rule discussed in Section 2.4, one can retrodict with certainty the outcome of a measurement of both $\hat{\sigma}_x$ and $\hat{\sigma}_z$ an an intermediate time. The authors then generalize this discussion to obtain perfect retrodiction of three components $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$ which is also known as the mean king's problem [58, 59].

We can have an analogous situation with continuous variables by considering squeezed Gaussian states which have a very low uncertainty on one quadrature and a correspondingly large noise on the conjugate one. Squeezed states of light can be produced with non-linear Hamiltonians or by monitoring a quadrature with homodyne detection.

We can imagine a situation where, for example, we prepare a past quantum state by monitoring quadrature $\hat{q}$ until a time t and we then measure $\hat{p}$, so that ρ has low uncertainty on the position and E has low uncertainty on the momentum. At the intermediate time t we can use Eqs. (5.16-5.17) to

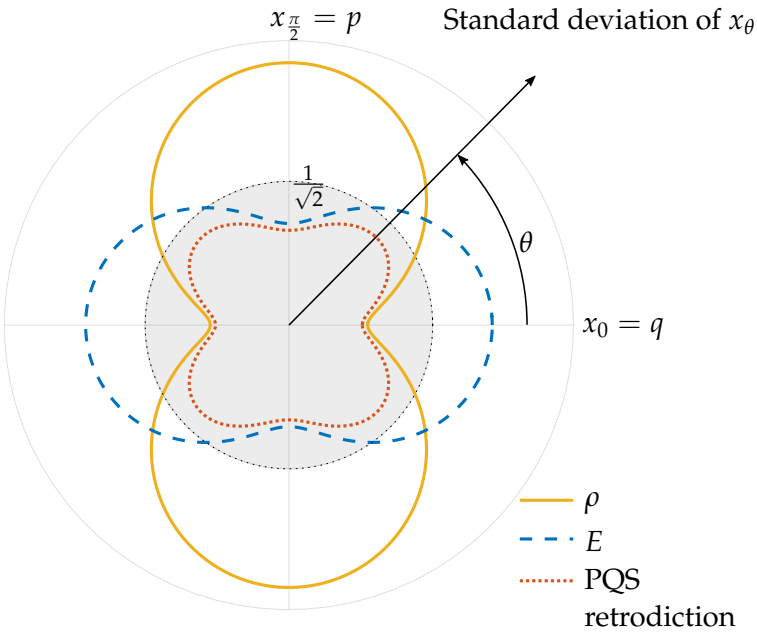

Figure 5.4 – Polar plot showing the standard deviation of different quadrature observables x_θ for a pure state ρ, squeezed along q, and an effect matrix E, squeezed along p. The usual elliptic phase space contours of Gaussian Wigner functions result in dumbbell shaped polar plots for ρ (solid) and E (dashed curve). The retrodiction, resulting from the combination of ρ and E, shows squeezing below the shot noise value of $\frac{1}{\sqrt{2}}$ for all quadrature angles, with minima in both the q and p directions (dotted curve).

retrodict the Gaussian past probability distribution for the measurement of any quadrature $\hat{x}_\theta$. In Fig. 5.4 we show the standard deviation of this distribution as a function of the direction θ defining the quadrature observables. We observe that the retrodicted uncertainty is smaller in all directions than the predicted uncertainty by the density matrix ρ alone, reflecting Eq. (5.16). We observe a Heisenberg "butterfly", reflecting the fact that the uncertainty may have minima along both the squeezed q and the squeezed p directions due to our probing of these observables before and after time t respectively.

5.4 Unobserved evolution until a final projective measurement

The effect matrix evolves backward in time from the condition at the final time T of being the identity operator. We can see from the backward master equation that, if no measurement is performed after time t, then for all the interval $[t, T]$ the effect matrix is the identity, and past quantum state is not different from the standard quantum mechanics formalism.

In this Section we study the situation where a system is subject to a projective measurement at a final time. It was shown in Ref. [60] that the past quantum state of a superconducting qubit, subject to a projective measurement at a final time T, differs appreciably from the usual exponential decay law, with consequences for retrodicted measurements on the system. Let us consider a similar example with a single oscillator mode, which at $t = 0$ occupies a coherent state $|\alpha\rangle$, and decays with rate Γ into the ground state. The master equation for the system is

$$\mathrm{d}\rho = \mathcal{D}[\sqrt{\Gamma}a]\rho\,\mathrm{d}t\,, \tag{5.20}$$

and the drift and diffusion matrices can be found to be $A = -\frac{\Gamma}{2}\mathbb{1}$ and $D = \Gamma\mathbb{1}$. This allows to calculate the evolution of the first and second moments of the past quantum state as

$$\frac{\mathrm{d}\langle \boldsymbol{r}\rangle}{\mathrm{d}t} = -\frac{\Gamma}{2}\langle \boldsymbol{r}\rangle, \qquad \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \Gamma(\mathbb{1} - \sigma), \tag{5.21}$$

$$\frac{\bar{\boldsymbol{r}}(t-dt) - \bar{\boldsymbol{r}}(t)}{\mathrm{d}t} = \frac{\Gamma}{2}\bar{\boldsymbol{r}}, \qquad \frac{\gamma(t-dt) - \gamma(t)}{\mathrm{d}t} = \Gamma(\mathbb{1} + \gamma). \tag{5.22}$$

The initial ρ is given as the coherent state $|\alpha\rangle = |\alpha_1 + i\alpha_2\rangle$, the evolution can be solved to obtain decaying mean values $\langle \boldsymbol{r}(t)\rangle = (\alpha_1, \alpha_2)^{\intercal}\sqrt{2}\,\mathrm{e}^{-\frac{\Gamma}{2}t}$ and constant covariance matrix, $\sigma(t) = \mathbb{1}$.

We assume that at time T the system is measured and projected onto the ground state, setting the boundary condition for the effect matrix and giving the solution

$$\bar{\boldsymbol{r}}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \gamma(t) = \mathbb{1}\left(2e^{\Gamma(T-t)} - 1\right). \tag{5.23}$$

This result is in agreement with an analysis in the Fock state basis given in Ref. [61].

By applying Eq. (5.18) for the retrodiction of position quadrature, we have that the past probability distribution is a Gaussian with the mean value

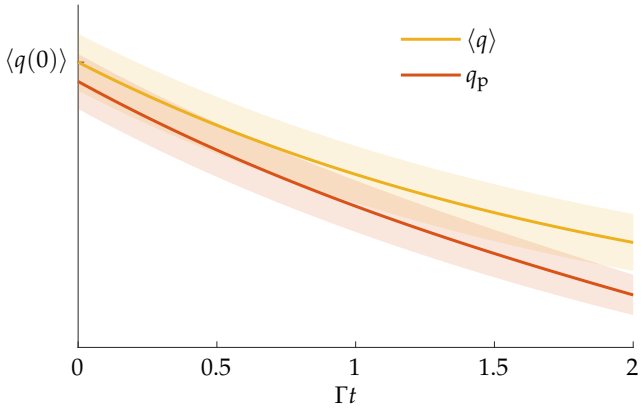

Figure 5.5 – Decay of a coherent state. The system is measured to be in the vacuum at the final time $T = 2\Gamma^{-1}$. The average position in the forward analysis (yellow) is a decaying coherent state. The past quantum state (red) gives a correction to both the mean value and the uncertainty of the position that is more important the closer we are to the final projective measurement. The uncertainty is given by the shaded area given by $\sqrt{\sigma_{qq}}$ and $\sqrt{\Delta_{\mathrm{p}}}$ respectively.

q_{p} and variance $\mathrm{Var}(q_{\mathrm{p}}) = \Delta(q_{\mathrm{p}})/2$ given by Eq. (5.18):

$$\begin{aligned} q_{\mathrm{p}}(t) &= \sqrt{2}\alpha_1 \left(e^{-\frac{\Gamma}{2}t} - \frac{1}{2} e^{-\Gamma\left(T-\frac{t}{2}\right)} \right), \\ \Delta(q_{\mathrm{p}}, t) &= 1 - \frac{1}{2} e^{-\Gamma(T-t)}. \end{aligned} \tag{5.24}$$

We can compare the forward prediction and the past retrodiciton for the position in Fig. 5.5.

It is clear with this example that even without continuously probing the system, but using only post-selection by a final measurement, the past quantum state can give a result significantly different from the conventional quantum state.

5.5 Continuous QND measurement

A quantum nondemolition (QND) measurement of an observable is attained when the uncertainty of the measured observable does not increase during

the subsequent evolution of the system, and successive measurements of the QND observable yield the same result. This concept was first studied in relation to the detection of gravitational waves [62, 63], where one of the main challenges is to increase the sensitivity of the measurement apparatus beyond the standard quantum limit [36, 64, 65]. Since then QND measurements have been studied and implemented in other experimental setups [28, 66–69] and have been studied also in discrete variable systems with application in quantum information [70].

In this Section we will apply past quantum state theory to a simple QND measurement setup.

Let us consider two harmonic oscillators: a system (s) and a meter (m), that interact with Hamiltonian

$$\hat{H} = g\hat{p}_{\mathrm{s}}\hat{p}_{\mathrm{m}}. \tag{5.25}$$

We assume that the system interacts only with the meter, while we have access to meter and we perform homodyne measurement on it so that the master equation for the composite system is given by

$$\mathrm{d}\rho = -i\big[\hat{H},\rho\big]\,\mathrm{d}t + \Gamma\mathcal{D}[\hat{a}_{\mathrm{m}}]\rho\,\mathrm{d}t + \sqrt{\eta\Gamma}\mathcal{H}[\hat{a}_{\mathrm{m}}]\rho\,\mathrm{d}W\,. \tag{5.26}$$

Through the interaction of the meter we effectively perform a QND measurement on $\hat{p}_{\mathrm{s}}$.

We define the vector $\hat{\boldsymbol{r}}^{\intercal} = (\hat{q}_{\mathrm{s}}, \hat{p}_{\mathrm{s}}, \hat{q}_{\mathrm{m}}, \hat{p}_{\mathrm{m}})$and we can give a Gaussian description of the monitored system. In particular we know that the evolution of σ and γ is deterministic and can be given by Eq. (C.16) and Eq. (C.27), namely

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \tilde{A}\sigma + \sigma\tilde{A}^{\intercal} + \tilde{D} - \sigma\tilde{B}\sigma, \qquad \frac{\mathrm{d}\gamma}{\mathrm{d}t} = -\tilde{A}\gamma - \gamma\tilde{A}^{\intercal} + \tilde{D} - \gamma\tilde{B}\gamma. \tag{5.27}$$

with the matrices calculated with Eqs. (C.17-C.19) to be

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 \\ 0 & g & \Gamma(\eta - \frac{1}{2}) & 0 \\ 0 & 0 & 0 & -\frac{\Gamma}{2} \end{pmatrix}, \qquad \begin{aligned} \tilde{D} &= \mathrm{diag}(0, 0, \Gamma(1-\eta), \Gamma), \\ \tilde{B} &= \mathrm{diag}(0, 0, \Gamma\eta, 0). \end{aligned} \tag{5.28}$$

It is clear that $\hat{p}_{\mathrm{s}}$ is coupled only with $\hat{q}_{\mathrm{m}}$ and we can limit ourselves to studying only the covariance matrix relative to this subsystem.

The differential Riccati equation for both the forward and backward evolution can be analytically solved by the matrix fraction decomposition method described in Appendix D.

We assume that the system is prepared with initial variance on $\hat{p}_s$ to be $\sigma_p/2 = \sigma_{22}(0)/2$ and the meter is in a minimum uncertainty state.

The uncertainty of the QND variable $\hat{p}_s$ in the forward evolution is then given by

$$\sigma_{22}(t) = \frac{\sigma_p}{1 + 4\eta g^2 \Gamma^{-2} \sigma_p \left(\Gamma t + 4\, e^{-\frac{\Gamma t}{2}} - e^{-\Gamma t} - 3 \right)}. \tag{5.29}$$

We can see that for large times the uncertainty on $\hat{p}_s$ can be reduced arbitrarily. It is possible to see that the uncertainty on the conjugate variable increases accordingly. The solution for the backward covariance matrix has a similar form.

The notable feature of the past estimation for the QND variable $\hat{p}_s$ is that the uncertainty can be calculated analytically to be constant at all times. Since $\hat{p}_s$ does not change during the natural evolution of the system, the uncertainty at the end of probing time and the one at any intermediate time (when we combine past and future measurements) have to coincide. The past quantum state formalism confirms this intuition.

5.6 Hybrid quantum-classical system

Related to measurements of QND variables we can consider the situation where a quantum system is used to measure a classical parameter.

We will consider, as an example, the situation where we gather information about a magnetic field by probing the spin of an atomic gas with a laser beam. This has been studied both theoretically and experimentally in Refs. [23, 71–75].

Let us consider a gas with N two-level atoms. Each atom is described by the vector of Pauli matrices $\boldsymbol{\sigma}_i$ and we can define the total effective spin operator as $\boldsymbol{J} = \frac{1}{2}\sum_i \boldsymbol{\sigma}_i$. Suppose that the system is prepared to have all the atomic spins polarized in the positive x direction so that initially $\langle J_x \rangle = \frac{N}{2}$. If we assume a very large number of atoms and that the they have only a weak interaction with the environment, then during the evolution of the system we can approximate $\hat{J}_x \approx \langle J_x \rangle$. Since $\left[\hat{J}_y, \hat{J}_z\right] = i\hat{J}_z$ we can redefine the variables

$$\hat{q} = \frac{\hat{J}_y}{\sqrt{\langle J_x \rangle}}, \qquad \hat{p} = \frac{\hat{J}_z}{\sqrt{\langle J_x \rangle}}, \tag{5.30}$$

that obey the canonical commutation relation $[\hat{q}, \hat{p}] = i$.

We assume that there is a constant magnetic field interacting with the atoms. This will cause a Larmor rotation of the atomic spins to precess

around the direction of the magnetic field. Let us choose the magnetic field to be in the y direction and have magnitude B. The interaction Hamiltonian can then be written as

$$\hat{H} = \gamma B \hat{q}_{\text{at}} \propto B \hat{J}_y, \tag{5.31}$$

where γ is the gyromagnetic ratio.

The atomic gas is also interacting with a laser beam that is used to probe the atoms via the interaction mediated by Faraday rotation, as is discussed in Ref. [23]. The outgoing light is measured with homodyne detection and this conditions the state of the atomic gas according to the discussion given in Section 3.5. In the limit of continuous monitoring we have that the evolution is described by the stochastic master equation

$$\mathrm{d}\rho = -i\mu[B\hat{q}, \rho]\,\mathrm{d}t + \kappa\mathcal{D}[\hat{p}]\rho\,\mathrm{d}t + \sqrt{\kappa}\mathcal{H}[\hat{p}]\rho\,\mathrm{d}W\,, \tag{5.32}$$

where $\mu = \gamma\sqrt{\langle J_x \rangle}$ and κ quantify the interaction strength between the atomic spins and the probe light. We can give therefore a Gaussian description of the system.

We include in the definition of the system the classical variable B, since it admits in principle a quantum description. The "classicalitity" comes from the fact that we assume the variable that is canonically conjugate to B has infinitely large uncertainty, so that B can be known as precisely as wanted, and the other variable does not interact with the system at all and can be neglected. The theory for Gaussian states applies completely by considering B as an operator that commutes with all the other variables of the system (which gives the suitable generalization for Eq. (3.19)).

If we define the vector of first moments as $\hat{\boldsymbol{r}}^{\intercal} = (\hat{q}, \hat{p}, B)$, then the evolution of the Gaussian description can be read from the master equation to be

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mu \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{aligned} \tilde{D} &= \mathrm{diag}(2\kappa, 0, 0), \\ \tilde{B} &= \mathrm{diag}(0, 2\kappa, 0). \end{aligned} \tag{5.33}$$

This can be solved by applying the method described in Appendix D to the 2×2 submatrix that is decoupled from the rest of the system. If we assume that the initial spin state is at minimum uncertainty and let $\sigma_B/2 = \sigma_{33}(0)/2$ be the initial variance on the magnetic field then the solution

for the covariance matrix is given by $\sigma = WU^{-1}$ with

$$W = \begin{pmatrix} 1 & -\mu\sigma_B t \\ 0 & \sigma_B \end{pmatrix}, \tag{5.34}$$

$$U = \begin{pmatrix} 1 + 2\kappa t & -\kappa\mu\sigma_B t^2 \\ \mu t + \kappa\mu t^2 & 1 - \frac{1}{3}\kappa\mu^2\sigma_B t^3 \end{pmatrix}. \tag{5.35}$$

Similarly the covariance matrix of the effect matrix of the system can be solved backward in time assuming that at the final time T we have a very large, but finite, uncertainty on both $\hat{p}$ and B that we indicate respectively with $\gamma_p = \gamma_{22}(T)$ and $\gamma_B = \gamma_{33}(T)$, obtaining an analytical form also for γ.

Our past estimate on the noise of the magnetic field can calculated by evaluating the relevant matrix element of $(\sigma^{-1} + \gamma^{-1})^{-1}$. We find that, when we include past quantum state theory, the past estimation for the uncertainty on B is constant at all time and, in the limit of infinitely large γ_B, it is given by the lowest uncertainty reached during the forward evolution, as shown in Fig. 5.6. Since B is effectively a QND variable, this is in agreement with the discussion in the previous Section.

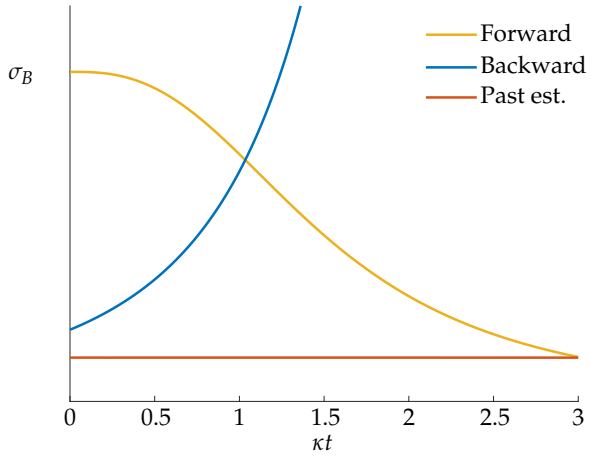

Figure 5.6 – Uncertainty on the magnetic field B for the density operator (yellow), the effect matrix (blue) and the past estimation (red). $\sigma_B/2$ is the initial variance on B. Probing of the atomic system contributes to the reduction in the uncertainty forward in time for ρ and backward in time for E. Since B is a constant classical variable it is intuitively clear that at the end of the experiment we can extrapolate the lowest uncertainty reached for all past times.

Chapter 6

State reconstruction and cooling of optomechanical systems

The field of optomechanics studies the interaction between optical fields and motion of massive mechanical degrees of freedom. A paradigmatic example is given by the light sustained by an optical cavity and the motion of a suspended end-mirror, that interact via radiation pressure, but many other setups have been studied recently, such as vibrating cantilevers or levitating nano-objects inside an optical cavity and nanofabricated optomechanical crystals [76]. This interaction has been used to cool mesoscopic objects to quantum ground state and to generate non-classical states such as squeezed or entangled states of these systems. These possibilities have attracted interest for both applicative and fundamental reasons [25].

In this Chapter we apply the Gaussian formalism that has been previously discussed to describe and analyze an optomechanical setup where we have as resonator a silica rod that can sustain both optical and mechanical whispering gallery modes (WGM).

Whispering gallery modes are named after the acoustic phenomenon that occurs inside some convex buildings where sound produced on one side can clearly be heard around the circumference. It was studied and explained by Rayleigh around the end of the 19th century when he found that sound can travel around a concave surface with a wave that is highly localized close to the surface.

An optical WGM is a wave traveling around the circumference of the silica cylinder by total internal reflection, while the mechanical WGM consists of material density waves at the surface. Both modes are confined in the

direction of the cylinder by rings cut on the surface of the resonator [77] and are respectively quantized as photons and phonons.

The optomechanical interaction between the two different types of resonances is mediated by electrostriction and photoelasticity [78]. Electrostriction is the compression or dilation of any dielectric material in the presence of an electric field[1], while photoelasticity is the change in optical properties of a medium due to the presence of mechanical pressure waves.

This interaction allows for Brillouin scattering, which is the inelastic scattering of light from material waves. The scattering can be differentiated into a Stokes process, where the material is heated and the outgoing photon is less energetic than the incoming photon, and an anti-Stokes process where the outgoing photon has a higher frequency than the incoming photon, draining energy from the material.

The setup is schematically represented in Fig. 6.1. We consider a pump laser that creates the optical WGM at frequency ω_{p} via evanescent coupling through a tapered optical fiber. This optical mode scatters off a co-rotating mechanical mode of frequency ω_{m} to an anti-Stokes mode and then we detect that signal at frequency

$$\omega_{\mathrm{aS}} = \omega_{\mathrm{p}} + \omega_{\mathrm{m}}. \tag{6.1}$$

In such a system the most studied process so far has been stimulated Brillouin scattering [79–82], which is a Stokes process where the pump and scattered optical mode generate at their beating frequency, through a self-consistent process, enough driving to excite the mechanical mode; for this reason stimulated scattering can occur only above a certain pump power. We, on the other hand, focus on spontaneous Brillouin scattering, where the pump mode scatters off from sound waves that are already thermally excited [78, 81].

6.1 Quantum model of the interaction

We briefly summarize the derivation of a quantum mechanical description of Brillouin scattering given in Ref. [80]. A more general approach can be found in Ref. [83].

[1] In electrostriction phenomena the external field induces a dipole in the dielectric material that interacts with said field. The strain is therefore proportional to the square of the polarization. We can contrast this with piezoelectric effect which is present only for specific classes of crystals, where there is a spontaneous dipole moment due to the asymmetry of the crystalline structure. This asymmetry causes a mechanical compression to generate a dipole moment that can interact with an external field. The strain is therefore linear in the polarization.

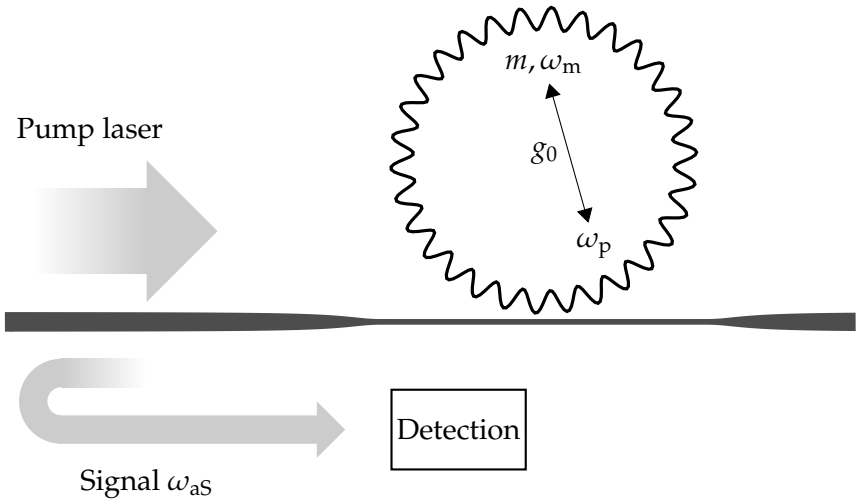

Figure 6.1 – Optomechanical system for spontaneous Brillouin scattering. The mechanical mode consists of a whispering gallery mode (WGM) characterized by effective mass m and frequency ω_m. The optical pump excites an optical WGM at frequency ω_p via evanescent coupling through a tapered optical fiber. The interaction between the two WGMs is mediated by electrostriction and is characterized by coupling strength g_0. The optical mode is scattered into an anti-Stokes WGM at a frequency ω_{aS}. The back-scattered anti-Stokes signal is subjected to detection.

We consider a longitudinal density fluctuation described by the displacement field u propagating in the z direction for a distance L and with a cross section of area A_m. Analogously, we call the cross section of the optical wave A_{opt}, and we assume that it is much smaller than A_m. The displacement field has two energy contributions:

- $\frac{1}{2}A_m T(\partial_z u)^2$, which is the elastic potential energy, with T the spring constant per area;
- $\frac{1}{2}A_m \rho \dot{u}^2$, which is the kinetic energy, and where ρ is the mass density.

The interaction energy mediated by electrostriction between the density

fluctuation and an electric field E is

$$H_{\text{int}} = \frac{1}{2} A_{\text{opt}} \gamma (\partial_z u) E^2, \qquad \text{where } \gamma = \rho \frac{\partial \epsilon}{\partial \rho}. \tag{6.2}$$

γ is the electrostrictive constant that relates changes in ϵ, the permittivity, and ρ, the density of the material.

We require periodic boundary conditions in the z direction and quantize the electric and the displacement fields as

$$\hat{E}_z = \sum_k E_k \left(\hat{a}_k \, \mathrm{e}^{ikz} + \text{h.c.} \right), \qquad \hat{u}_z = \sum_k u_k \left(\hat{b}_k \, \mathrm{e}^{ikz} + \text{h.c.} \right). \tag{6.3}$$

By considering all the energy contributions and enforcing energy conservation in the anti-Stokes process as in Eq. (6.1) and momentum conservation as $k_{\text{aS}} = k_{\text{p}} + k_{\text{p}}$, we obtain the full Hamiltonian of the system as

$$\hat{H} = g_0 \left(\hat{a}_{\text{aS}}^\dagger \hat{b} \hat{a}_{\text{p}} + \hat{a}_{\text{aS}} \hat{b}^\dagger \hat{a}_{\text{p}}^\dagger \right) + \omega_{\text{aS}} \hat{a}_{\text{aS}}^\dagger \hat{a}_{\text{aS}} + \omega_{\text{m}} \hat{b}^\dagger \hat{b} + \omega_{\text{p}} \hat{a}_{\text{p}}^\dagger \hat{a}_{\text{p}}. \tag{6.4}$$

Going into the appropriate interaction picture, the Hamiltonian can be reduced to

$$\hat{H} = g_0 \left(\hat{a}_{\text{aS}}^\dagger \hat{b} \hat{a}_{\text{p}} + \hat{a}_{\text{aS}} \hat{b}^\dagger \hat{a}_{\text{p}}^\dagger \right) - \Delta \hat{b}^\dagger \hat{b}, \tag{6.5}$$

with the detuning defined as

$$\Delta = \omega_{\text{aS}} - \omega_{\text{p}} - \omega_{\text{m}}. \tag{6.6}$$

We then assume the parametric approximation of the pump being in a strong coherent state, so that we can make the substitution $\hat{a}_{\text{p}} \to \alpha = |\alpha| \, \mathrm{e}^{i\varphi}$. If we assume $\varphi = 0$, we obtain the effective Hamiltonian with a beam-splitter type interaction

$$\hat{H} = g(\hat{a}_{\text{aS}} \hat{b} + \hat{a}_{\text{aS}} b^\dagger) - \Delta \hat{b}^\dagger \hat{b}, \tag{6.7}$$

with $g = g_0 |\alpha|$ the enhanced optomechanical coupling strength.

6.2 Open system dynamics

At room temperature we can consider the mechanical mode interacting with a thermal bath with average phonon number n_{th}, and the anti-Stokes mode interacting with a zero-temperature bath, so that the Lindblad master equation of the system can be found to be

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i\left[\hat{H}, \rho\right] + \gamma(n_{\text{th}} + 1)\mathcal{D}[\hat{b}]\rho + \gamma n_{\text{th}} \mathcal{D}[\hat{b}^\dagger]\rho + \kappa \mathcal{D}[\hat{a}_{\text{aS}}]\rho \equiv \mathcal{L}\rho, \tag{6.8}$$

where γ and κ are the damping rates of the mechanical and optical modes respectively, and the Hamiltonian is given by Eq. (6.7).

This is exactly the type of master equation discussed in Chapter 4 and we know therefore that a Gaussian description is possible. Let us define the canonical quadrature vector as $\hat{\boldsymbol{r}}^\intercal = (\hat{q}_\mathrm{m}, \hat{p}_\mathrm{m}, \hat{q}_\mathrm{aS}, \hat{p}_\mathrm{aS})$. The evolution of the system is given by the following dynamical equations for the first and second moments,

$$\frac{\mathrm{d}\langle \boldsymbol{r}\rangle}{\mathrm{d}t} = A\langle \boldsymbol{r}\rangle, \tag{6.9}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = A\sigma + \sigma A^\intercal + D, \tag{6.10}$$

with the drift and diffusion matrix given respectively by

$$A = \begin{pmatrix} -\frac{\gamma}{2} & -\Delta & & g \\ \Delta & -\frac{\gamma}{2} & -g & \\ & g & -\frac{\kappa}{2} & \\ -g & & & -\frac{\kappa}{2} \end{pmatrix}, \tag{6.11}$$

$$D = \mathrm{diag}\left((2n_\mathrm{th}+1)\gamma, (2n_\mathrm{th}+1)\gamma, \kappa, \kappa\right). \tag{6.12}$$

The steady state of the system can be analytically found by setting the left hand side of Eq. (6.10) equal to 0 (in which case the equation is called an algebraic Riccati equation). As mentioned in Section 3.5, the covariance matrix of the mechanical mode is given by the sub-covariance matrix of the composite system, so the average number of phonons in the mechanical mode can easily be calculated from the steady state covariance matrix with

$$\begin{aligned} n_\mathrm{m} &= \frac{\mathrm{tr}[\sigma_\mathrm{m}] - 2}{4} \\ &= n_\mathrm{th}\gamma \frac{4g^2(\gamma+\kappa) + \kappa(4\Delta^2 + (\gamma+\kappa)^2)}{4g^2(\gamma+\kappa)^2 + \gamma\kappa(4\Delta^2 + (\gamma+\kappa)^2)}, \end{aligned} \tag{6.13}$$

where σ_m is the mechanical sub-covariance matrix.

Equation (6.13) gives the performance of the sideband cooling of the mechanical mode induced by the beam splitter interaction with the optical mode.

6.2.1 Cooling via measurement

In optomechanical systems the cooling and reconstruction of the state of macroscopic resonators is an important achievement that is necessary for the emergence and study of quantum effects.

For this purpose, different schemes have been proposed where probing of the system has a fundamental role. In particular homodyne detection has been employed to implement feedback control with continuous detection that enhances the optomechanical interaction [84, 85]; to achieve state preparation in pulsed regime [86, 87]; to conditionally prepare Gaussian [26] and non-Gaussian states [88]; and to cool to the quantum ground state and stabilize mechanical motion [89].

As discussed in Chapter 5, monitoring a Gaussian system induces a nonlinear evolution of the covariance matrix that can only reduce its eigenvalues. From Eq. (6.13) we see that measuring the system can reduce the number of excitations in the mode, thereby effectively cooling the system.

In using Eq. (6.13) to calculate the number of excitations we are assuming that the first moments of the system are zero. As a matter of fact, the probing of the system will induce stochastic evolution of the first moments. However, with the Gaussian formalism that we are using, the first moments at every time are known, which means that at least in principle we can use the knowledge obtained from the measurement data to actively control the first moments of the system.

We study in this Section the cooling rate in the situation where homodyne or heterodyne detection is performed on the anti-Stokes mode. We assume for simplicity to have zero detuning in Eq. (6.8), that is $\Delta = 0$. This allows us to obtain an analytical solution for the steady state covariance matrix where the $(\hat{p}_{\mathrm{m}}, \hat{q}_{\mathrm{aS}})$ and $(\hat{q}_{\mathrm{m}}, \hat{p}_{\mathrm{aS}})$ subsystems are decoupled.

Homodyne measurement In the case of homodyne detection on the anti-Stokes mode the master equation is

$$\mathrm{d}\rho = \mathcal{L}\rho + \sqrt{\kappa\eta}\,\mathcal{H}[\hat{a}_{\mathrm{aS}}]\rho\,\mathrm{d}W\,, \tag{6.14}$$

where η is the efficiency of the measurement. In this case the steady state covariance matrix gives for the number of mechanical excitations

$$\begin{aligned} n_{\mathrm{m}} = \frac{1}{4}\Bigg(&\frac{2n_{\mathrm{th}}\gamma(4g^2+\kappa^2+\kappa\gamma)}{(4g^2+\kappa\gamma)(\kappa+\gamma)} \\ &+\frac{(Q_2(\eta)-\kappa-\gamma)(2Q_1(\eta)+\kappa^2+\gamma^2-\gamma Q_2(\eta)+\kappa Q_2(\eta))}{16g^2\eta\kappa}\Bigg)\,, \end{aligned} \tag{6.15}$$

where

$$Q_1(\eta) = \sqrt{(4g^2+\kappa\gamma)^2+32n_{\mathrm{th}}g^2\kappa\gamma\eta}, \tag{6.16}$$

$$Q_2(\eta) = \sqrt{\kappa^2+\gamma^2-8g^2+2Q_1(\eta)}. \tag{6.17}$$

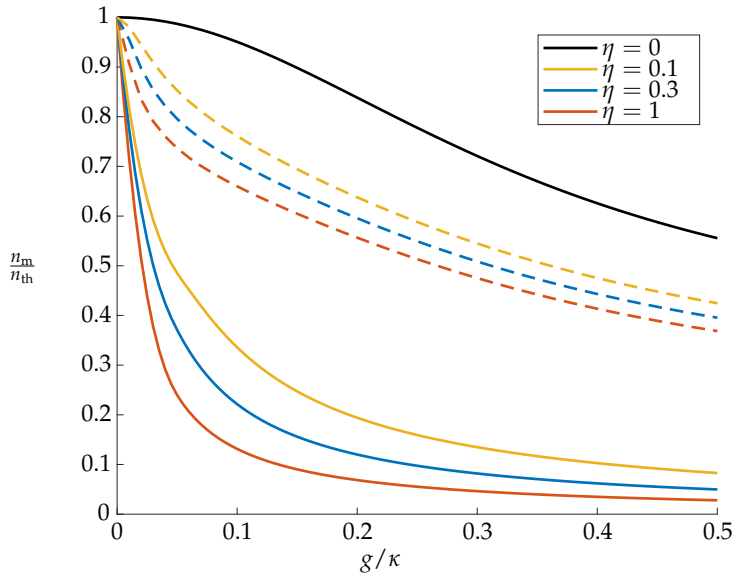

Figure 6.2 – Cooling rate for the mechanical mode for homodyne (dashed) and heterodyne (solid) detection. The unmonitored sideband cooling is obtained for $\eta = 0$. The numerical parameters are $\gamma = 0.5\kappa$, $n_{\rm th} = 600$.

Heterodyne measurement As discussed in Section 4.2, in the case of heterodyne detection the stochastic master equation reads

$$\mathrm{d}\rho = \mathcal{L}\rho + \sqrt{\frac{\kappa\eta}{2}}\mathcal{H}[\hat{a}_{\rm aS}]\rho\,\mathrm{d}W_x + \sqrt{\frac{\kappa\eta}{2}}\mathcal{H}[-i\hat{a}_{\rm aS}]\rho\,\mathrm{d}W_y\,. \tag{6.18}$$

The steady state number of mechanical excitation is given in this case by

$$n_{\rm m} = \frac{1}{4}\frac{\left(Q_2(\frac{\eta}{2}) - \kappa - \gamma\right)\left(2Q_1(\frac{\eta}{2}) + \kappa^2 + \gamma^2 + (\kappa-\gamma)Q_2(\frac{\eta}{2})\right)}{4g^2\eta\kappa}. \tag{6.19}$$

In Fig. 6.2 we compare the cooling rate for heterodyne and homodye detection. We see that heterodyne detection has a better performance even for relatively low efficiency and optomechanical coupling.

6.3 Analysis of experimental data

As discussed in the previous Section, the cooling of the mechanical mode is effective only if we have the conditioned first moment evolution of the

system. We now discuss the quantum state reconstruction for the system conditioned on experimental measured current, and the attainable cooling of the mechanical resonator with heterodyne detection.

We describe the heterodyne detection as a rotating homodyne. The master equation therefore reads

$$\mathrm{d}\rho = \mathcal{L}\rho + \sqrt{\kappa\eta}\mathcal{H}[\hat{a}_{\mathrm{aS}}\,\mathrm{e}^{-i\omega_{\mathrm{het}}t}]\rho\,\mathrm{d}W\,, \tag{6.20}$$

where the Wiener noise is related to the measurement current by

$$\begin{aligned}\mathrm{d}y(t) =& \sqrt{\kappa\eta}\langle a_{\mathrm{aS}}\,\mathrm{e}^{-i\omega_{\mathrm{het}}t} + a_{\mathrm{aS}}^{\dagger}\,\mathrm{e}^{i\omega_{\mathrm{het}}t}\rangle\,\mathrm{d}t + \mathrm{d}W(t) = \\ =& \sqrt{2\kappa\eta}\langle q_{\mathrm{aS}}\cos(\omega_{\mathrm{het}}t) + p_{\mathrm{aS}}\sin(\omega_{\mathrm{het}}t)\rangle\,\mathrm{d}t + \mathrm{d}W(t)\,.\end{aligned} \tag{6.21}$$

Since the evolution of the covariance matrix is deterministic, it can be found by simply solving the differential Riccati equation. Due to the modeling of the monitoring, the system will not reach a proper steady state, but will have a "squeezed" quadrature that is rotating in the phase space with frequency ω_{het}.

For the same reason the cooling that has been studied in the previous Section will constitute a lower bound for the experimental situation, and we see that it can be reached in the limit of very large ω_{het}.

6.3.1 Normalization of measurement current with limited detector bandwidth

The experimental measurement current $\tilde{y}(t)$ can be normalized by calibrating with the vacuum signal. In principle the experimental signal from the vacuum state is rescaled white noise and we can find the correct normalization factor by ensuring that the Wiener noise has the correct standard deviation, namely $y(t) = \alpha\tilde{y}(t)$ with

$$\mathrm{std}(y(t)) = \mathrm{d}t^2\,. \tag{6.22}$$

We see, however, in Fig. 6.3 that the detector has limited bandwidth and has the expected flat white noise spectrum for low frequencies. This does not constitute a problem in the reconstruction of the state of the system, since the experimental setup has relevant dynamics only below the threshold frequency. The purpose of the normalization is then to have the correct measurement signal $y(t)$ for the relevant low frequencies.

More formally, given a signal $f(t)$ and its Fourier transform, $F(\omega)$, let us define the quantity

$$A_f(\omega) = \sum_{|\omega'|<\omega} \left|F(\omega')\right|^2. \tag{6.23}$$

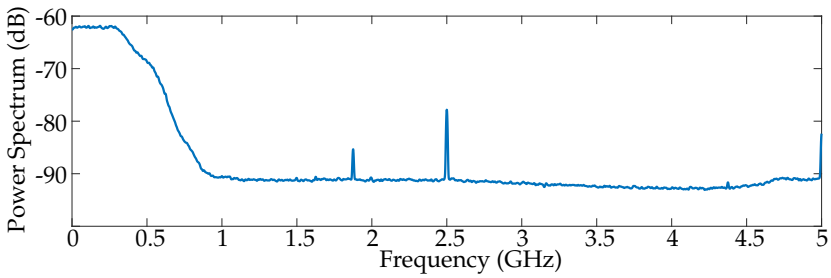

Figure 6.3 – Experimental power spectrum for vacuum state. We see that the detector has limited bandwidth and the spectrum is flat only for frequencies that are lower than about 300 MHz.

$A_f(\omega)$ is essentially the power content of the signal for frequencies lower than ω.

By Parseval's theorem we can translate the normalization condition given in Eq. (6.22) into a condition in the frequency domain. If f is truly white noise then we know that F is a constant, and therefore $A(\omega)$ is a linear function of ω. We need to normalize the experimental signal in such a way that the low frequency components match correctly with white noise. As can be seen in Fig. 6.4 the experimental vacuum signal is linear only for low frequencies and the correct normalization is obtained by rescaling the experimental signal to match a simulated flat-spectrum white noise.

The stochastic evolution of the system can then be calculated by Eq. (5.9) to have the first moments of the full optomechanical system at every time. A sample of this evolution is shown in Fig. 6.5 where the first moments evolve stochastically and are conditioned on the measurement record, while the evolution of the second moments is deterministic and, as anticipated, oscillates at the heterodyne frequency.

In order to study the effect of the limited bandwidth of the detector, we compare the first moment evolution obtained by conditioning on white signal from an ideal detector with the evolution from the filtered signal of an imperfect detector. In Fig. 6.6 a sample of the evolution of $\langle q_{\mathrm{m}} \rangle$ is shown. We see that the high frequency components of the noise do not change significantly the reconstructed mean value and the dynamic is correctly captured up to small corrections.

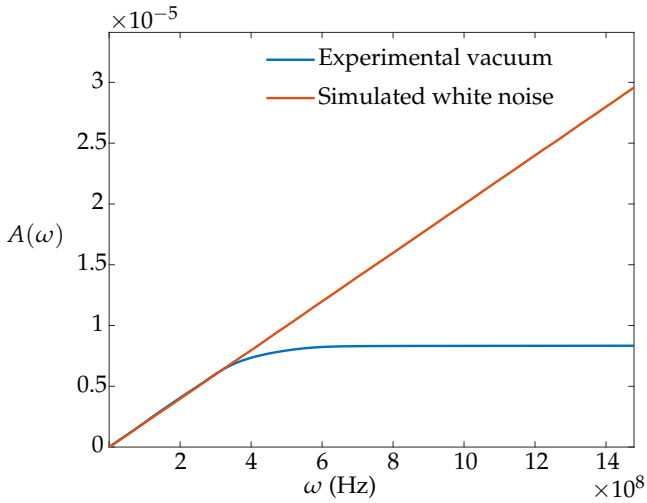

Figure 6.4 – Power spectrum content for correctly normalized experimental vacuum signal. The white noise to which it is compared is simulated to have the correct normalization given in Eq. (6.22).

Physical quantity	Symbol	Value
Mechanical frequency	ω_{m}	$2\pi \times 11$ GHz
Anti-Stokes frequency	ω_{aS}	$2\pi \times 193.5796$ THz
Pump frequency	ω_{p}	$2\pi \times 193.5685$ THz
Mechanical linewidth	γ	$2\pi \times 45$ MHz
Anti-Stokes linewidth	κ	$2\pi \times 91$ MHz
Pump linewidth	κ_{p}	$2\pi \times 13.3$ MHz
Coupling strenght	g	$2\pi \times 6.1$ MHz
Detuning	Δ	$2\pi \times -17$ MHz
Heterodyne frequency	ω_{het}	$2\pi \times 154$ MHz
Total efficiency	η	0.31
Thermal excitation	n_{th}	553
Sampling frequency	f_{s}	10 GHz

Table 6.1 – Experimental values considered.

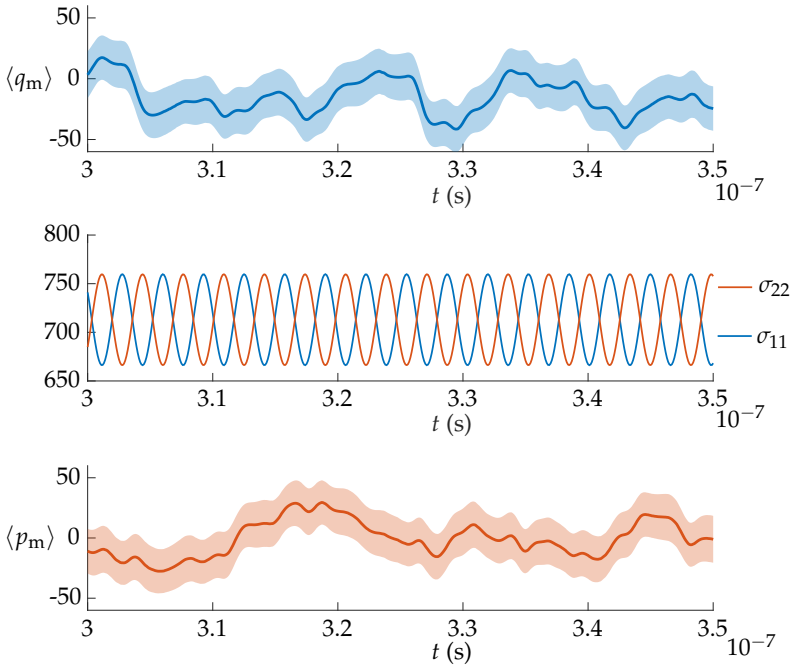

Figure 6.5 – Time evolution for mechanical position (top panel) and momentum (bottom panel) between 300 ns and 350 ns. The shaded area corresponds to the standard deviation derived from the covariance matrix. The middle panel shows the corresponding covariance matrix elements that oscillate at the heterodyne frequency. Experimental parameters are in Table 6.1.

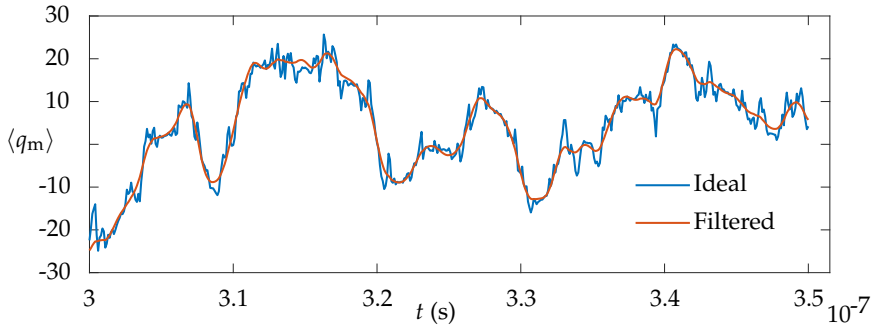

Figure 6.6 – Mechanical position dynamics simulated with an ideal detector with infinite bandwidth (blue) and the same noise filtered to have the same power spectrum as in Fig. 6.3 (red). The imperfect detector correctly captures the dynamics of the system and the corrections are purely due to noise. The experimental parameters that are used in the simulation are in Table 6.1.

6.3.2 Cooling

The analysis of the cooling of the system does not necessitate the experimental measurement current. In Fig. 6.7 we show the steady state variance of the mechanical position for different coupling strengths[2] in the monitored and unmonitored case. As previously discussed, this variance is directly related to the average number of excitations in the mechanical mode. The cooling induced by heterodyne detection is then clear.

In phase space representation the unmonitored state is represented by a large Gaussian distribution centered at 0. The unmonitored state is, on the other hand, a narrower Gaussian distribution with a mean value that has a non-trivial stochastic trajectory. If we take the average of the conditioned Gaussians, we expect to recover the original broad distribution. This is confirmed in Fig. 6.7, where it is clear that the sum of the width of the narrow monitored Gaussian distribution and the variance of the phase space trajectory gives the unmonitored uncertainty of the system.

[2]To be precise, in the system at hand g and Δ are not independent from each other so every data point has different values of both interaction strength and detuning.

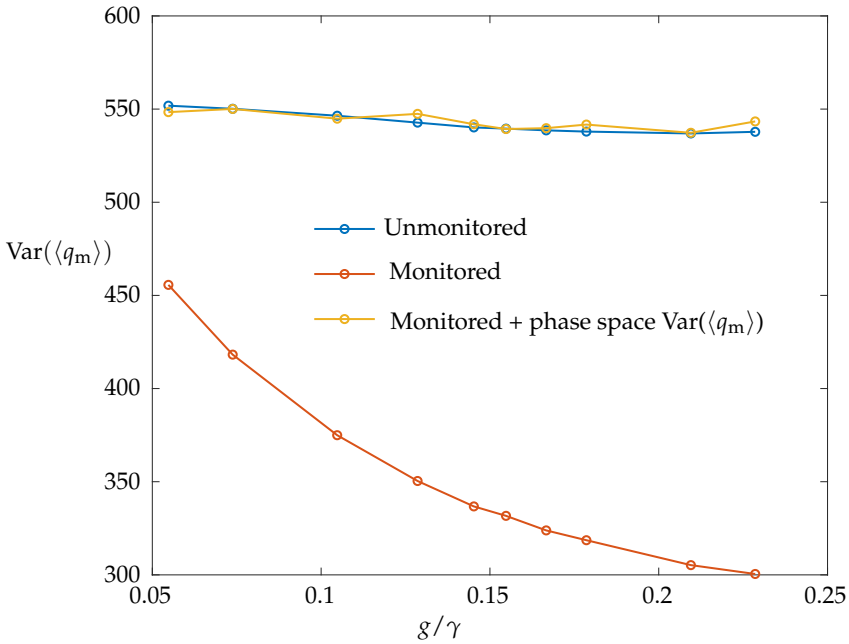

Figure 6.7 – Variance of the mechanical position for the unmonitored system (blue) and the monitored system (red). Since the variance is oscillating at the heterodyne frequency, here we show the stationary value obtained by averaging over one heterodyne period. The monitored system has a non-trivial phase space trajectory. The variance of this trajectory accounts for the difference between the monitored and unmonitored situation (yellow).

Appendix A

Phase space with real variables

We follow the convention of Ref. [33] and define the displacement operator for real parameters as

$$\hat{D}(\boldsymbol{r}) = \mathrm{e}^{i\boldsymbol{r}^{\intercal}\Omega\hat{\boldsymbol{r}}}\,. \tag{A.1}$$

The correspondence with the displacement operator with complex variable is $\hat{D}(\alpha) = \hat{D}(-\boldsymbol{r})$ where $\boldsymbol{r}^{\intercal} = (q,p) = \sqrt{2}(\mathrm{Re}[\alpha], \mathrm{Im}[\alpha])$. In particular for a single mode

$$\hat{D}(\boldsymbol{r}) = \mathrm{e}^{i(q\hat{p}+p\hat{q})} = \mathrm{e}^{iq\hat{p}}\,\mathrm{e}^{-ip\hat{q}}\,\mathrm{e}^{-\frac{i}{2}qp} = \mathrm{e}^{-ip\hat{q}}\,\mathrm{e}^{iq\hat{p}}\,\mathrm{e}^{\frac{i}{2}qp}\,. \tag{A.2}$$

The composition of displacement operators obeys the rule

$$\hat{D}(\boldsymbol{r})\hat{D}(\boldsymbol{s}) = \hat{D}(\boldsymbol{r}+\boldsymbol{s})\,\mathrm{e}^{-\frac{i}{2}\boldsymbol{r}^{\intercal}\Omega\boldsymbol{s}}\,. \tag{A.3}$$

The characteristic function is defined as

$$\chi(\boldsymbol{r}) = \mathrm{Tr}\left[\hat{D}(-\boldsymbol{r})\rho\right], \tag{A.4}$$

while the Fourier-Weyl relation is

$$\hat{O} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \mathrm{d}^{2n}\boldsymbol{r}\,\chi(\boldsymbol{r})\hat{D}(\boldsymbol{r}), \tag{A.5}$$

and the orthogonality of displacement operators reads

$$\mathrm{Tr}\left[\hat{D}(\boldsymbol{r})\hat{D}(-\boldsymbol{s})\right] = (2\pi)^n\delta(\boldsymbol{r}-\boldsymbol{s}). \tag{A.6}$$

The Wigner function is defined as the Fourier transform of the characteristic function

$$W(\boldsymbol{s}) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} \mathrm{d}^{2n}\boldsymbol{r}\, \mathrm{e}^{-i\boldsymbol{r}^{\intercal}\Omega^{\intercal}\boldsymbol{s}}\, \chi(\boldsymbol{r}), \tag{A.7}$$

with inverse transform

$$\chi(\boldsymbol{s}) = \int_{\mathbb{R}^{2n}} \mathrm{d}^{2n}\boldsymbol{r}\, \mathrm{e}^{i\boldsymbol{s}^{\intercal}\Omega^{\intercal}\boldsymbol{r}}\, W(\boldsymbol{r}). \tag{A.8}$$

The trace of the product of two operators can be calculated as integrals in phase space with

$$\begin{aligned} \mathrm{Tr}\left[\hat{O}_1\hat{O}_2\right] &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \mathrm{d}^{2n}\boldsymbol{r}\, \chi_{\hat{O}_1}(\boldsymbol{r})\chi_{\hat{O}_2}(-\boldsymbol{r}) \\ &= (2\pi)^n \int_{\mathbb{R}^{2n}} \mathrm{d}^{2n}\boldsymbol{r}\, W_{\hat{O}_1}(\boldsymbol{r}) W_{\hat{O}_2}(\boldsymbol{r}). \end{aligned} \tag{A.9}$$

The delta function can be expanded as

$$\int_{\mathbb{R}^{2n}} \mathrm{d}^{2n}\boldsymbol{r}\, \mathrm{e}^{i\boldsymbol{r}^{\intercal}\Omega\boldsymbol{s}} = (2\pi)^{2n}\delta(\boldsymbol{s}). \tag{A.10}$$

A quadrature operator acting on the density matrix corresponds to the following in phase space

$$\begin{aligned} \hat{q}_j\rho &\to \left(q_j + \frac{i}{2}\partial_{p_j}\right) W_\rho(r), \qquad & \hat{p}_j\rho &\to \left(p_j - \frac{i}{2}\partial_{q_j}\right) W_\rho(r), \\ \rho\hat{q}_j &\to \left(q_j - \frac{i}{2}\partial_{p_j}\right) W_\rho(r), \qquad & \rho\hat{p}_j &\to \left(p_j + \frac{i}{2}\partial_{q_j}\right) W_\rho(r). \end{aligned} \tag{A.11}$$

Appendix B

Input-output theory

In the present Chapter we will follow the analysis of [90] to study an optical system interacting with a heat bath and derive the master equation as a specific example of Lindblad master equation (4.1). More detailed analysis can be found in [91, 91, 92]. We will drop the hat notation for operators since it causes no ambiguity. The Hamiltonian for the system, the bath and their interaction is

$$H = H_{\text{sys}} + H_{\text{B}} + H_{\text{int}}, \tag{B.1}$$

$$H_{\text{B}} = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega b^{\dagger}(\omega) b(\omega), \tag{B.2}$$

$$H_{\text{int}} = i \int_{-\infty}^{\infty} \mathrm{d}\omega\, \kappa(\omega) \left(b^{\dagger}(\omega) c - c^{\dagger} b(\omega) \right), \tag{B.3}$$

where c is an operator for the system, $b(\omega)$ are the annihilation operators for the bath that satisfy the commutation relation

$$\left[b(\omega), b^{\dagger}(\omega') \right] = \delta(\omega - \omega'), \tag{B.4}$$

and $\kappa(\omega)$ is the interaction strength of the linear coupling between the system and the bath.

In Heisenberg's picture we can find the equation of motion of the annihilation operators of the bath $b(\omega)$ and any operator of the system a as

$$\dot{b}(\omega) = i[H, b(\omega)] = -i\omega b(\omega) + \kappa(\omega) c, \tag{B.5}$$

$$\dot{a} = i\left[H_{\text{sys}}, a\right] + \int \mathrm{d}\omega\, \kappa(\omega) (b^{\dagger}(\omega)[a, c] - \left[a, c^{\dagger}\right] b(\omega) \tag{B.6}$$

We define an in field by

$$b_{\text{in}} = \frac{1}{2\pi} \int \mathrm{d}\omega \, \mathrm{e}^{-i\omega(t-t_0)} \, b_0(\omega), \tag{B.7}$$

which satisfies commutation relation

$$\left[b_{\text{in}}(t), b_{\text{in}}^{\dagger}(t') \right] = \delta(t - t'), \tag{B.8}$$

and choose the interaction $\kappa(\omega) = \sqrt{\gamma/(2\pi)}$ which describes damping into a Markovian bath. By combining the above one can obtain the quantum Langevin equation

$$\dot{a} = i\big[H_{\text{sys}}, a\big] - \left(\left[a, c^{\dagger} \right] \left(\frac{\gamma}{2} c + \sqrt{\gamma} b_{\text{in}}(t) \right) - \left(\frac{\gamma}{2} c^{\dagger} + \sqrt{\gamma} b_{\text{in}}^{\dagger}(t) \right) [a, c] \right). \tag{B.9}$$

In the simplest case where the coupling is through the system's annihilation operator and therefore $c = a$ we get the Langevin equation for a damped system

$$\dot{a} = i\big[H_{\text{sys}}, a\big] - \frac{\gamma}{2} a - \sqrt{\gamma} b_{\text{in}}(t). \tag{B.10}$$

Appendix C

Derivation of evolution of a Gaussian state

In the following we present the derivation of the forward and backward evolution of a Gaussian past quantum state following the discussion in the appendix of [1].

C.1 Forward evolution

We consider the stochastic master equation for n bosonic modes described by $\boldsymbol{r}^\intercal = (q_1, p_1, \dots, q_n, p_n)$ discussed in Chapter 5

$$\mathrm{d}\rho = -i\big[\hat{H}, \rho\big]\,\mathrm{d}t + \sum_h \mathcal{D}[\hat{c}_h]\,\mathrm{d}t + + \sum_h \sqrt{\eta_h}\mathcal{H}[\hat{c}_h]\rho\,\mathrm{d}W_h(t)\,. \tag{C.1}$$

We assume quadratic Hamiltonian $\hat{H}$ and linear coupling with the environment, so that if we define $\hat{\boldsymbol{c}}^\intercal = (c_1, \dots, c_m)$ we have

$$\hat{H} = \frac{1}{2}\hat{\boldsymbol{r}}^\intercal R \hat{\boldsymbol{r}}, \qquad \hat{\boldsymbol{c}} = \widetilde{C}\hat{\boldsymbol{r}}. \tag{C.2}$$

The evolution of an element $\langle r_i \rangle$ of the first moment vector is obtained with $\mathrm{d}\langle r_i \rangle = \mathrm{Tr}\,[\hat{r}_i\,\mathrm{d}\rho]$. We discuss the calculation by considering the three parts of the master equation, the Hamiltonian, dissipation and measurement contribution.

The evolution induced by the Hamiltonian is

$$\begin{aligned}
-i\,\mathrm{Tr}\left[\left[\hat{H},\rho\right]\hat{r}_i\right] &= -i\,\mathrm{Tr}\left[\rho\left[\hat{r}_i,\hat{H}\right]\right] = \\
&= -\frac{i}{2}\sum_{jk} R_{jk}\langle\left[r_i,r_jr_k\right]\rangle = \\
&= -\frac{i}{2}\sum_{jk} R_{jk}\left(\langle r_j[r_i,r_k]\rangle + \langle\left[r_i,r_j\right]r_k\rangle\right) = \\
&= -\frac{i}{2}\sum_{jk} R_{jk}\,\mathrm{Tr}\left[i\rho\left(\hat{r}_j\Omega_{ik}+\hat{r}_k\Omega_{ij}\right)\right] = \\
&= -\sum_{jk}\Omega_{ik}R_{kj}\langle r_j\rangle,
\end{aligned} \tag{C.3}$$

where we used the symmetry of matrix R.

We write explicitly the definition of the dissipation channels as

$$\hat{c}_h = \sum_k \widetilde{C}_{hk}\hat{r}_k, \qquad \hat{c}_h^\dagger = \sum_l \widetilde{C}^*_{hl}\hat{r}_l, \tag{C.4}$$

then if we consider the action of a single dissipation channel $\hat{c}_h$

$$\begin{aligned}
&\mathrm{Tr}\left[\left(\hat{c}_h\rho\hat{c}_h^\dagger - \frac{1}{2}\left\{\hat{c}_h^\dagger\hat{c}_h,\rho\right\}\right)\hat{r}_i\right] = \\
&= \sum_{kl}\mathrm{Tr}\left[\widetilde{C}_{hk}\widetilde{C}^*_{hl}\left(\hat{r}_l\hat{r}_i\hat{r}_k - \frac{1}{2}\{\hat{r}_i,\hat{r}_l\hat{r}_k\}\right)\rho\right] = \\
&= \frac{i}{2}\sum_{kl}\Omega_{il}\left(\widetilde{C}^*_{hk}\widetilde{C}_{hl} - \widetilde{C}_{hk}\widetilde{C}^*_{hl}\right)\langle r_k\rangle.
\end{aligned} \tag{C.5}$$

The contribution of all the dissipation terms is therefore

$$\begin{aligned}
&\sum_{hkl}\Omega_{il}\frac{1}{2i}\left(\widetilde{C}^*_{hl}\widetilde{C}_{hk} - \widetilde{C}_{hl}\widetilde{C}^*_{hk}\right)\langle r_k\rangle = \\
&= \sum_{kl}\Omega_{il}\frac{1}{2i}\left([\widetilde{C}^\dagger\widetilde{C}]_{lk} - [\widetilde{C}^\intercal\widetilde{C}^*]_{lk}\right)\langle r_k\rangle = \\
&= \sum_{kl}\Omega_{il}\left[\mathrm{Im}[\widetilde{C}^\dagger\widetilde{C}]\right]_{lk}\langle r_k\rangle,
\end{aligned} \tag{C.6}$$

where we used the properties $\widetilde{C}^*_{hl} = [\widetilde{C}^\dagger]_{lh}$ and $\widetilde{C}_{hl} = [\widetilde{C}^\intercal]_{lh}$.

Finally for the measurement contribution let us define the real and imaginary part of $\widetilde{C}$ as $a_{hk} = \mathrm{Re}[\widetilde{C}_{hk}]$, $b_{hk} = \mathrm{Im}[\widetilde{C}_{hk}]$, then we have for the measurement on mode $\hat{c}_h$

$$\begin{aligned} &\mathrm{Tr}\left[(\hat{c}_h - \langle c_h\rangle)\rho\hat{r}_i + \rho(\hat{c}_h^\dagger - \langle c_h^\dagger\rangle)\hat{r}_i\right] = \\ &= \sum_k a_{hk}\left(\langle r_i r_k\rangle + \langle r_k r_i\rangle - 2\langle r_i\rangle\langle r_k\rangle\right) + i b_{hk}\left(\langle r_i r_k\rangle - \langle r_k r_i\rangle\right) = \\ &= \sum_k a_{hk}\sigma_{ik} - b_{hk}\Omega_{ik} = \sigma_{ik}\,\mathrm{Re}[\widetilde{C}^\intercal]_{kh} - \Omega_{ik}\,\mathrm{Im}[\widetilde{C}^\intercal]_{kh}. \end{aligned} \tag{C.7}$$

To conclude, the stochastic evolution of the first moment in vector form can be written as

$$\mathrm{d}\langle \boldsymbol{r}\rangle = A\langle \boldsymbol{r}(t)\rangle + (\sigma(t)\,\mathrm{Re}[\widetilde{C}]^\intercal - \Omega\,\mathrm{Im}[\widetilde{C}]^\intercal)\sqrt{\boldsymbol{\eta}}\,\mathrm{d}\boldsymbol{W}(t)\,, \tag{C.8}$$

where the drift matrix is defined as

$$A = \Omega\left(R + \mathrm{Im}[\widetilde{C}^\dagger\widetilde{C}]\right), \tag{C.9}$$

and we define the efficiency matrix as $\boldsymbol{\eta} = \mathrm{diag}(\eta_1, \ldots, \eta_m)$, the vector of Wiener noise as $\mathrm{d}\boldsymbol{W}(t)^\intercal = (\mathrm{d}W_1(t), \ldots, \mathrm{d}W_m(t))$, which is related to the vector of measurement noise $\mathrm{d}\boldsymbol{y}(t)^\intercal = (\mathrm{d}y_1(t), \ldots, \mathrm{d}y_m(t))$ by

$$\mathrm{d}\boldsymbol{y}(t) = \sqrt{\boldsymbol{\eta}}\left(\widetilde{C} + \widetilde{C}^*\right)\langle \boldsymbol{r}(t)\rangle\,\mathrm{d}t + \mathrm{d}\boldsymbol{W}(t)\,. \tag{C.10}$$

When studying the evolution of the covariance matrix we need to correctly apply Itô calculus rules and therefore we need to consider also second order differentials and use

$$\begin{aligned} \mathrm{d}\sigma_{ij} = &\,\mathrm{Tr}\left[\mathrm{d}\rho\left(\hat{r}_i\hat{r}_j + \hat{r}_j\hat{r}_i\right)\right] + \\ &- 2\,\mathrm{d}\langle r_i\rangle\,\langle r_j\rangle - 2\langle r_i\rangle\,\mathrm{d}\langle r_j\rangle - 2\,\mathrm{d}\langle r_i\rangle\,\mathrm{d}\langle r_j\rangle\,. \end{aligned} \tag{C.11}$$

A notable property of Gaussian states is that under master equation C.1 the evolution of the covariance matrix does not depend on the measurement outcome. This is due to the specifically Gaussian property that all higher moments of the Wigner function can be expressed in terms of first and second order moments by Wick's theorem. We will therefore consider only the deterministic terms fo the evolution in the following.

The Hamiltonian contribution to $\mathrm{Tr}\big(\mathrm{d}\rho\, r_i r_j\big)$ is

$$\begin{aligned} -i\,\mathrm{Tr}\left[\left[\hat{H},\rho\right]\hat{r}_i\hat{r}_j\right] &= -\frac{i}{2}\sum_{kl} R_{kl}\,\mathrm{Tr}\left[\rho\left[\hat{r}_i\hat{r}_j,\hat{r}_k\hat{r}_l\right]\right] = \\ &= \sum_{kl} R_{kl}\left(\langle r_i r_k\rangle\Omega_{jl} + \langle r_k r_j\rangle\Omega_{il}\right), \end{aligned} \tag{C.12}$$

while the dissipation gives

$$\begin{aligned}&\mathrm{Tr}\left[\left(\hat{c}_h\rho\hat{c}_h^\dagger-\frac{1}{2}\left\{\hat{c}_h^\dagger\hat{c}_h,\rho\right\}\right)\hat{r}_i\hat{r}_j\right]=\\&=\frac{i}{2}\sum_{kl}\widetilde{C}_{hk}\widetilde{C}_{hl}^*(\langle r_ir_k\rangle\Omega_{lj}-\langle r_jr_k\rangle\Omega il+\\&\qquad-\langle r_lr_i\rangle\Omega_{kj}+\langle r_lr_j\rangle\Omega_{ik}).\end{aligned}\tag{C.13}$$

By combining all the terms in Eq. (C.11), the evolution of the covariance matrix can be written as the Riccati matrix

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}t}=&A\sigma+\sigma A^\intercal+D\\&-2(\sigma\,\mathrm{Re}[\widetilde{C}]^\intercal-\Omega\,\mathrm{Im}[\widetilde{C}]^\intercal)\boldsymbol{\eta}(\sigma\,\mathrm{Re}[\widetilde{C}]^\intercal-\Omega\,\mathrm{Im}[\widetilde{C}]^\intercal)^\intercal,\end{aligned}\tag{C.14}$$

where the diffusion matrix is defined as

$$D=-2\Omega\,\mathrm{Re}[\widetilde{C}^\dagger\widetilde{C}]\Omega\tag{C.15}$$

Sometimes it is more useful to group terms that depend on σ in the same way and we get the alternative expression

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}=\tilde{A}\sigma+\sigma\tilde{A}^\intercal+\tilde{D}-\sigma\tilde{B}\sigma,\tag{C.16}$$

where the matrices are defined as

$$\tilde{A}=A+2\Omega\,\mathrm{Im}[\widetilde{C}]^\intercal\boldsymbol{\eta}\,\mathrm{Re}[\widetilde{C}],\tag{C.17}$$

$$\tilde{D}=D+2\Omega\,\mathrm{Im}[\widetilde{C}]^\intercal\boldsymbol{\eta}\,\mathrm{Im}[\widetilde{C}]\Omega,\tag{C.18}$$

$$\tilde{B}=2\,\mathrm{Re}[\widetilde{C}]^\intercal\boldsymbol{\eta}\,\mathrm{Re}[\widetilde{C}].\tag{C.19}$$

C.2 Backward evolution

The derivation of the backward evolution for a Gaussian effect matrix follows the same lines of the forward evolution and indeed many parts are very similar. We will therefore in the following focus on the parts that are most different.

The backward master equation is

$$\begin{aligned}\mathrm{d}E=i\left[\hat{H},E\right]\mathrm{d}t+\sum_h\left(\mathcal{D}^\dagger[\hat{c}_h]E-\left(\overline{c_hc_h^\dagger}-\overline{c_h^\dagger c_h}\right)E\right)\mathrm{d}t+\\+\sum_h\sqrt{\eta_h}\mathcal{H}[\hat{c}_h^\dagger]E\,\mathrm{d}s_h(t-\mathrm{d}t)\,,\end{aligned}\tag{C.20}$$

where we define the stochastic quantity $\mathrm{d}s_h$ as

$$\mathrm{d}s_h(t-\mathrm{d}t) = \mathrm{d}y_h(t-\mathrm{d}t) - \sqrt{\eta_h}\,\mathrm{Tr}\left[E\left(\hat{c}_h + \hat{c}_h^\dagger\right)\right]\mathrm{d}t\,. \tag{C.21}$$

Whenever it is not explicitly specified all quantities are to be evaluated at time t.

We can also define the vector vector $\mathrm{d}\boldsymbol{s}^\intercal = (\mathrm{d}s_1, \ldots, \mathrm{d}s_m)$ and it is related to the measurement current as

$$\mathrm{d}\boldsymbol{y}(t-\mathrm{d}t) = \sqrt{\boldsymbol{\eta}}\left(\widetilde{C} + \widetilde{C}^*\right)\overline{\boldsymbol{r}(t)}\,\mathrm{d}t + \mathrm{d}\boldsymbol{s}(t-\mathrm{d}t)\,. \tag{C.22}$$

By comparing the backward master equation C.20 with the forward master equation C.1 we can see that the Hamiltonian contribution to the evolution differs only by a sign, while the the measurement terms change by $\hat{c}_h \to \hat{c}_h^\dagger$. We will then focus on the calculation of the evolution for the dissipation superoperators.

We can calculate the dissipation term as

$$\begin{aligned}&\mathrm{Tr}\left[\left(\hat{c}_h^\dagger E\hat{c}_h - \frac{1}{2}\left\{\hat{c}_h^\dagger\hat{c}_h, E\right\} - (\overline{c_h c_h^\dagger} - \overline{c_h^\dagger c_h})E\right)\hat{r}_i\right] = \\ &= -\frac{i}{2}\sum_{kl}\Omega_{il}\left(\widetilde{C}^*_{hk}\widetilde{C}_{hl} - \widetilde{C}_{hk}\widetilde{C}^*_{hl}\right)\overline{r_k}.\end{aligned} \tag{C.23}$$

The above considerations give the evolution of the first moments of the effect matrix as

$$\mathrm{d}\bar{\boldsymbol{r}} = -A\overline{\boldsymbol{r}(t)}\,\mathrm{d}t + (\gamma\,\mathrm{Re}[\widetilde{C}]^\intercal + \Omega\,\mathrm{Im}[\widetilde{C}]^\intercal)\sqrt{\boldsymbol{\eta}}\,\mathrm{d}\mathbf{s}(t-\mathrm{d}t)\,. \tag{C.24}$$

In studying the evolution of the covariance matrix γ we have that the stochastic contributions cancel also for the backward master equation C.20. Again, we show only the contribution from the dissipation part:

$$\begin{aligned}&\mathrm{Tr}\left[\left(\hat{c}_h^\dagger E\hat{c}_h - \frac{1}{2}\left\{\hat{c}_h^\dagger\hat{c}_h, E\right\} - (\overline{c_h c_h^\dagger} - \overline{c_h^\dagger c_h})E\right)\hat{r}_i\hat{r}_j\right] = \\ =&\frac{i}{2}\sum_{kl}\widetilde{C}_{hk}\widetilde{C}^*_{hl}(-\overline{r_k r_i}\Omega_{lj} + \overline{r_k r_j}\Omega_{il} + \\ &\qquad + \overline{r_i r_l}\Omega_{kj} - \overline{r_j r_l}\Omega_{ik}).\end{aligned} \tag{C.25}$$

Eventually we can derive the backward Riccati equation as

$$\begin{aligned}\frac{\mathrm{d}\gamma}{\mathrm{d}t} =& - A\gamma - \gamma A^\intercal + D \\ &- 2(\gamma\,\mathrm{Re}[\widetilde{C}]^\intercal + \Omega\,\mathrm{Im}[\widetilde{C}]^\intercal)\boldsymbol{\eta}(\gamma\,\mathrm{Re}[\widetilde{C}]^\intercal + \Omega\,\mathrm{Im}[\widetilde{C}]^\intercal)^\intercal.\end{aligned} \tag{C.26}$$

By using Eqs (C.17-C.19) the backward evolution of γ can be also written as

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = -\tilde{A}\gamma - \gamma\tilde{A}^{\intercal} + \tilde{D} - \gamma\tilde{B}\gamma. \tag{C.27}$$

Appendix D

Matrix fraction decomposition for differential Riccati equation

The properties of the Riccati equation have been extensively studied in the context of classical control theory, both in its differential form [93] and its algebraic form [94]. We briefly present a method for the analytical solution of a differential Riccati equation called matrix fraction decomposition that reduces the non-linear equation into a linear system [95]. Consider the matricial equation

$$\dot{X} = AX + XA^{\intercal} - XBX + D, \tag{D.1}$$

with initial condition $X(0) = X_0$: we look for a solution of the form $X = WU^{-1}$. By using the derivative of the inverse of a matrix as

$$\frac{\mathrm{d}U^{-1}}{\mathrm{d}t} = -U^{-1}\dot{U}U^{-1}, \tag{D.2}$$

we have that the Riccati equation is equivalent to the two following coupled linear differential equations

$$\dot{W} = AW + DU, \tag{D.3}$$
$$\dot{U} = BW - A^{\intercal}U. \tag{D.4}$$

As initial condition we can choose $W(0) = X_0$ and $U(0) = \mathbb{1}$.

Appendix E

Miscellanea

- Baker-Campbell-Hausdorff formula

$$e^{\hat{A}}\,\hat{B}\,e^{-\hat{A}} = \hat{B} + \left[\hat{A},\hat{B}\right] + \frac{1}{2!}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right] + \cdots + \frac{1}{n!}\left[\hat{A},\left[\ldots,\left[\hat{A},\hat{B}\right]\right]\right] + \ldots \tag{E.1}$$

  If $\left[\hat{A},\hat{B}\right]$ commutes with both $\hat{A}$ and $\hat{B}$

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}\,e^{\hat{B}}\,e^{-\frac{1}{2}\left[\hat{A},\hat{B}\right]} \tag{E.2}$$

- Normally and anti-normally ordered displacement operator

$$\hat{D}(\alpha) = \exp\left[\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right] = e^{-\frac{1}{2}|\alpha|^2}\,e^{\alpha\hat{a}^\dagger}\,e^{-\alpha^*\hat{a}} = e^{\frac{1}{2}|\alpha|^2}\,e^{-\alpha^*\hat{a}}\,e^{\alpha\hat{a}^\dagger}. \tag{E.3}$$

- Given a symmetric and positive definite $n \times n$ matrix A and a $n \times 1$ vector $\boldsymbol{B}$

$$\int_{\mathbb{R}^n} d^n\boldsymbol{r}\, e^{-\boldsymbol{r}^\intercal A\boldsymbol{r}+\boldsymbol{B}^\intercal\boldsymbol{r}} = \sqrt{\frac{\pi^n}{\mathrm{Det}\left[A\right]}}\, e^{\frac{1}{4}\boldsymbol{B}^\intercal A^{-1}\boldsymbol{B}} \tag{E.4}$$

- For a a positive number we have the relation

$$\int_{-\infty}^{\infty} dx\, x^2\, e^{-ax^2} = \frac{\sqrt{\pi}}{2}a^{-\frac{3}{2}}. \tag{E.5}$$

- Given $f(\beta)$ function of a complex variable α, the complex Fourier transform of f is defined as [30]

$$g(\alpha) = \int_{\mathbb{C}} \frac{d^2\beta}{\pi}\, e^{\alpha\beta^* - \alpha^*\beta} f(\beta). \tag{E.6}$$

  With this definition the inverse Fourier transform has the same form.

- Delta expansion

$$\pi^{2n}\delta(\boldsymbol{\alpha}) = \int_{\mathbb{C}^n} \mathrm{d}^{2n}\boldsymbol{\beta}\, \mathrm{e}^{\boldsymbol{\beta}^\dagger\boldsymbol{\alpha}-\boldsymbol{\alpha}^\dagger\boldsymbol{\beta}} \tag{E.7}$$

$$(2\pi)^{2n}\delta(\boldsymbol{s}) = \int_{\mathbb{R}^{2n}} \mathrm{d}^{2n}\boldsymbol{r}\, \mathrm{e}^{i\boldsymbol{r}^\intercal\Omega\boldsymbol{s}}\,. \tag{E.8}$$

$$2\pi\delta(t) = \int \mathrm{d}k\, \mathrm{e}^{ikt} \tag{E.9}$$

- Blockwise matrix inversion, given a $n \times n$ matrix A, $m \times m$ matrix B and $n \times m$ matrix C, then

$$\begin{pmatrix} A & C \\ C^\intercal & B \end{pmatrix}^{-1} = \begin{pmatrix} \left(A - CB^{-1}C^\intercal\right)^{-1} & A^{-1}C\left(C^\intercal A^{-1}C - B\right)^{-1} \\ \left(C^\intercal A^{-1}C - B\right)^{-1} C^\intercal A^{-1} & \left(B - C^\intercal A^{-1}C\right)^{-1} \end{pmatrix} \tag{E.10}$$

- Woodbury matrix identity, given a $n \times n$ matrix A, $m \times m$ matrix B, $n \times m$ matrix U, $m \times n$ matrix V the

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U\left(C^{-1} + VA^{-1}U\right)^{-1} VA^{-1}. \tag{E.11}$$

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