Crystalline beam emulations in a pulse-excited linear Paul trap

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This paper considers a pulsed voltage excitation of the quadrupole electrodes of a linear Paul ion trap. The transverse dynamics of ions in this time-varying electric field is analogous to that of charged particles in the strong focusing magnetic lattice of a storage ring. By laser cooling ions stored in a pulse-excited linear Paul trap theoretical results on the stability of crystalline ion beams in storage rings can be tested. The stability of ion motion in a pulse-excited trap is derived in (q,a)-parameter formalism and we show where in (q,a) space to expect the formation of Coulomb crystals according to the theory of crystalline ion beams. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1355024]

I. INTRODUCTION

The analogy between the effects of magnetic and electric quadrupole fields on charged particles is a subject of standard textbooks on charged particle beams (see, e.g., Refs. 1 and 2). As it turns out, motion in either magnetic or electric quadrupole fields of periodic strength variation in space or time, s, is governed by Hill's type of equations, i.e., equations of the form

$$\frac{d^2u}{ds^2}+[\lambda-Q(s)]u=0, \tag{1}$$

where λ is a constant and $Q(s)$ is a real, piecewise continuous function of period π. In a recent paper, Okamoto and Tanaka[3] pointed out that this parallel in the dynamics can be used to study beam halo formation (as observed for high intensity beams in accelerators with magnetic structures) in an electric trap. The idea was emphasized even more by Davidson *et al.*,[4] who proposed using a Paul trap configuration to simulate intense non-neutral beam propagation over large distances through a periodic focusing quadrupole magnetic field.

In this paper, we propose to go one step further and exploit the analogy between the transverse dynamics of a storage ring and of a pulse-excited linear Paul trap in that theoretical results on the stability of crystalline ion beams can be tested. Crystalline energetic ion beams[5] have not been observed so far in any storage ring even at very low beam temperatures (e.g., in laser cooling experiments with space-charge dominated beams[6]). However, some ordering effects in cold stored ion beams have been reported (see Ref. 7, and references therein). The lack of crystallization, as routinely observed in ion traps,[8–10] is probably due to the fact that the maintenance conditions for a beam crystal as predicted by theory have not been fulfilled so far in storage ring experiments.[11] Some of these results putting severe restrictions on, e.g., the phase advance per unit cell of the storage ring lattice can be tested experimentally in a linear Paul ion trap as outlined in this paper. A trap for experiments along this line using laser cooled $^{24}Mg^+$ ions is currently under construction in our laboratory in Aarhus. A similar experiment with the same goal has been initiated in Munich by Schätz *et al.*[12] As this experiment uses a ring trap in which the ions are to be accelerated, the problem of shear due to bending and the need for a suitably graded cooling force arise (see, e.g., Ref. 13, and references therein). Investigations of this kind might provide information on similar effects in a realistic storage ring environment. However, the scheme using a linear Paul trap benefits from the fact that the crystal interacts solely with the focusing structure. Hence, the stability problems regarding focusing can be addressed separately.

II. STRONG FOCUSING IN MAGNETIC STORAGE RINGS AND ELECTRIC QUADRUPOLE TRAPS

In a typical storage ring or synchrotron charged, energetic particles are made to circulate in a plane by the Lorentz force of magnetic dipole fields. Transverse confinement along the ideal orbit is obtained by using a periodic arrangement of quadrupole fields. The magnetic field in Cartesian coordinates (x,y,z) of a perfect quadrupole of strength g can be derived from a scalar potential $V(x,y)=gxy$ as $\mathbf{B}=-\nabla V$. For a particle of charge e and mass m with a velocity $\mathbf{v}=v\hat{\mathbf{z}}$ in the z direction taken to be along the ideal orbit, the Lorentz force $\mathbf{F}=e\mathbf{v}\times\mathbf{B}$ is

$$\mathbf{F}=-evgx\hat{\mathbf{x}}+evgy\hat{\mathbf{y}}. \tag{2}$$

For positively charged particles (i.e., $e>0$) and $g>0$, this force acts focusing in the x direction and defocusing in the y direction and gives rise to the equations of motions inside a magnetic quadrupole:

$$\ddot{u}+K_u u=0, \quad u=x,y, \tag{3}$$

where $K_{x,y}=\pm evg/m$. The principle of alternating gradient focusing (AG focusing) or so-called strong focusing[14] relies on the fact that by combining two quadrupoles into a doublet with the second having opposite polarity of the first (i.e., $g\rightarrow -g$) net focusing in both transverse dimensions (x,y) can be obtained by proper choice of quadrupole separation.

The linear Paul trap is an offspring of the well-known electric quadrupole mass filter as proposed by Paul and

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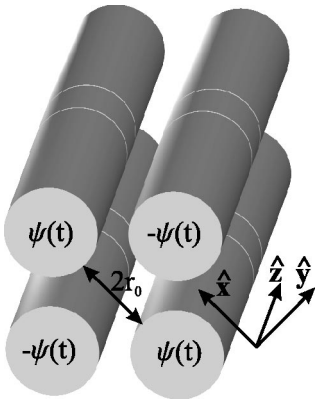

FIG. 1. Electric quadrupole configuration. A time-varying voltage $\pm\psi(t)$ is applied as shown. When operated as a linear Paul trap each electrode is sectioned into three and a dc voltage U_0 is applied to the eight end pieces in addition to $\psi(t)$ or $-\psi(t)$.

Steinwedel.[15] The latter essentially consists of four parallel cylindrical electrodes arranged in the way as shown in Fig. 1 and with voltages $\pm\psi$ applied to the rods as indicated. This configuration gives rise to the electric potential $\Phi(x,y)=\psi(x^2-y^2)/r_0^2$, where r_0 is the inscribed radius of the interelectrode space. Charged particles in the interelectrode space obey exactly the same transverse equations of motion as derived for a charged particle traveling through a magnetic quadrupole Eq. (3), where we now have $K_{x,y}=\pm 2em\psi/r_0^2$. Likewise, for positively charged particles and $\psi>0$ we get a focusing effect in the x direction and a defocusing effect in the y direction. To obtain transverse confinement in both directions, the idea is to use a periodic, time-dependent potential $\psi(t)$. This is very analogous to the spatially varying quadrupole potential as seen by a charged particle traveling through the periodic lattice of a storage ring with magnetic AG confinement. The conventional choice is a sinusoidal excitation

$$\psi(t)=U-V\cos\omega t. \tag{4}$$

Introducing reduced parameters

$$\begin{aligned}&\xi=\omega t/2,\quad a_x=8eU/mr_0^2\omega^2,\quad q_x=4eV/mr_0^2\omega^2,\\&a_y=-8eU/mr_0^2\omega^2,\quad q_y=-4eV/mr_0^2\omega^2,\end{aligned} \tag{5}$$

the equations of motion for the x and y directions can be transformed into the canonical form of Mathieu's equation:

$$\frac{d^2u}{d\xi^2}+(a_u-2q_u\cos 2\xi)u=0,\quad u=x,y. \tag{6}$$

The properties of Mathieu's equation have been recorded in detail (see, e.g., Ref. 16, and references therein). Here, we only note that the (q_u,a_u) plane is divided into stable and unstable regions by characteristic curves (see Fig. 2). For (q_u,a_u) in a stable region, solutions to Mathieu's equations are bound whereas in an unstable region the solutions tend to infinity with time. To have overall transverse stability (i.e., in both transverse dimensions x and y), we must demand that both (q_x,a_x) and (q_y,a_y) belong to stable regions of Eq. (6).

The choice of a sinusoidally time-dependent potential is by no means unique. The arbitrary nature of the scalar potential was emphasized by Richards *et al.*, who investigated the possibilities of using a rectangularly time-varying potential[17]

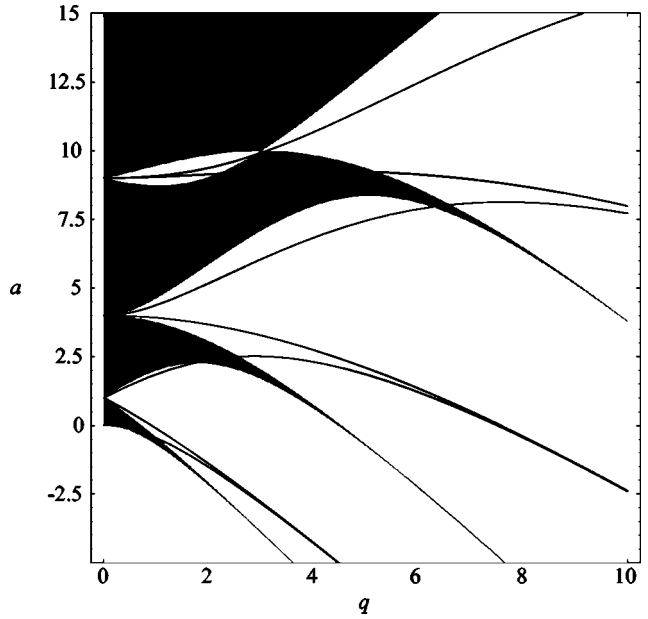

FIG. 2. (q,a)-stability diagrams. The lines bound the stability regions of Eq. (6), i.e., the case of sinusoidal excitation. The shaded areas show the stability regions of Eq. (9) when $\tau=1/2$ for pulsed excitation.

$$\psi(t)=U-VS_\delta(t), \tag{7}$$

where $S_\delta(t)$ is a rectangular function with duty cycle δ. In this paper, we consider a pulsed voltage excitation of the quadrupole electrodes having a wave form cycle period T, and pulse duration τT, on the specific form

$$\begin{aligned}&\psi(t)=U-VP_\tau(t),\\&P_\tau(t)=\begin{cases}1 & \text{if}\quad |t|\leqslant\tau T/2\\ 0 & \text{if}\quad \tau T/2<|t|\leqslant(1-\tau)T/2\\ -1 & \text{if}\quad (1-\tau)T/2<|t|\leqslant T/2,\end{cases}\\&P_\tau(t+T)=P_\tau(t).\end{aligned} \tag{8}$$

Defining here the angular repetition frequency to be $\omega=2\pi/T$ and using the transformations Eq. (5) the equations of motion for pulsed excitation appear as

$$\frac{d^2u}{d\xi^2}+[a_u-2q_u\tilde{P}_\tau(\xi)]u=0,\quad u=x,y, \tag{9}$$

where $\tilde{P}_\tau(\xi)$ (see Fig. 3) is the obvious transformation of $P_\tau(t)$ to the reduced parameter space

$$\begin{aligned}&\tilde{P}_\tau(\xi)=\begin{cases}1 & \text{if}\quad |\xi|\leqslant\dfrac{1}{2}\tau\pi\\ 0 & \text{if}\quad \dfrac{1}{2}\tau\pi<|\xi|\leqslant\dfrac{1}{2}(1-\tau)\pi\\ -1 & \text{if}\quad \dfrac{1}{2}(1-\tau)\pi<|\xi|\leqslant\dfrac{1}{2}\pi,\end{cases}\\&\tilde{P}_\tau(\xi+\pi)=P_\tau(\xi).\end{aligned} \tag{10}$$

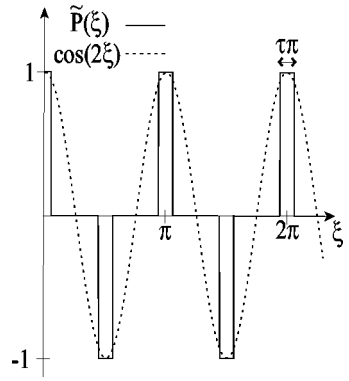

FIG. 3. The pulsed wave form $\tilde{P}(\xi)$ given by Eq. (10).

The time-varying pulsed electric field emulates the transverse dynamics of charged particles in the spatially varying magnetic field of a strong focusing lattice and we may apply theory developed for the latter to describe the stability of Eq. (9). As the coefficient $K_u(\xi) \equiv a_u - 2q_u \tilde{P}_\tau(\xi)$ is a piecewise constant function, a solution to Eq. (9) may be propagated from a vector containing the initial conditions using matrix multiplication

$$\begin{bmatrix} u(\xi) \\ u'(\xi) \end{bmatrix} = M_u(\xi|\xi_0) \begin{bmatrix} u(\xi_0) \\ u'(\xi_0) \end{bmatrix}, \tag{11}$$

where we have adopted notation from Ref. 18. In Eq. (11) $u(\xi_0)$ and $u'(\xi_0)$ are initial values at $\xi = \xi_0$ and $M_u(\xi|\xi_0)$ is the 2×2 overall transfer matrix from ξ_0 to ξ which can be calculated by successive multiplication of the transfer matrices for all the intervals of constant K_u between ξ_0 and ξ (see the Appendix).

The solutions to Eq. (9) will be stable if $|\mathrm{Tr}\, M_u(\xi|\xi + \pi)| < 2$ and unstable if $|\mathrm{Tr}\, M_u(\xi|\xi + \pi)| > 2$.[18] Thus, for a given τ we may find corresponding values of a and q leading to stable (single particle) motion and plot this as a stability diagram. An example of a stability diagram when $\tau = 1/2$ (i.e., the special case of a rectangular waveform) is shown in Fig. 2. In Fig. 4, the development of the first two stability regions for pulsed voltage excitation is shown when the pulse duration is decreased from $\tau = 0.5$ to $\tau = 0.05$

III. EMULATION OF A BEAM CRYSTAL IN A LINEAR PAUL TRAP

As described previously, we may obtain confinement in the two transverse directions of a pulse-excited electric quadrupole. This mimics the transverse dynamics of an AG magnetic lattice. To study crystalline stability in a stationary reference frame, such as the electric quadrupole shown in Fig. 1, we furthermore need axial confinement. In this respect, the physical situation resembles that of a bunched beam of particles rather than a coasting beam. Axial confinement can be achieved by sectioning each of the quadrupole rods into three as shown in Fig. 1 and applying a dc voltage U_0 to the eight end pieces in addition to $\psi(t)$ or $-\psi(t)$. This gives rise to a static potential having a saddle point at the center of the trap and we approximate this with a harmonic potential[19]

$$\phi_s = \eta U_0 [z^2 - \tfrac{1}{2}(x^2 + y^2)], \tag{12}$$

where η is geometric factor. By careful choice of design parameters of the trap, the harmonic approximation will indeed be valid. We note that the axially confining force arising from the static potential Eq. (12) is accompanied by a radially defocusing force in accordance with Earnshaw's theorem. This harmonic defocusing force may be accounted for by including it in the a_u parameter of Eq. (9). In fact, by choosing $U = 0$ in Eq. (8) this will be the only contribution to the a parameter

$$a_u = -\frac{4\eta e U_0}{m\omega^2}, \tag{13}$$

and we note that in this case $a_u < 0$ always.

So far, we have only dealt with the stability of a single particle or rather a particle in a plasma sufficiently dilute that the Coulomb interaction between particles can be neglected. By applying laser cooling to, e.g., $^{24}Mg^+$ ions, which has

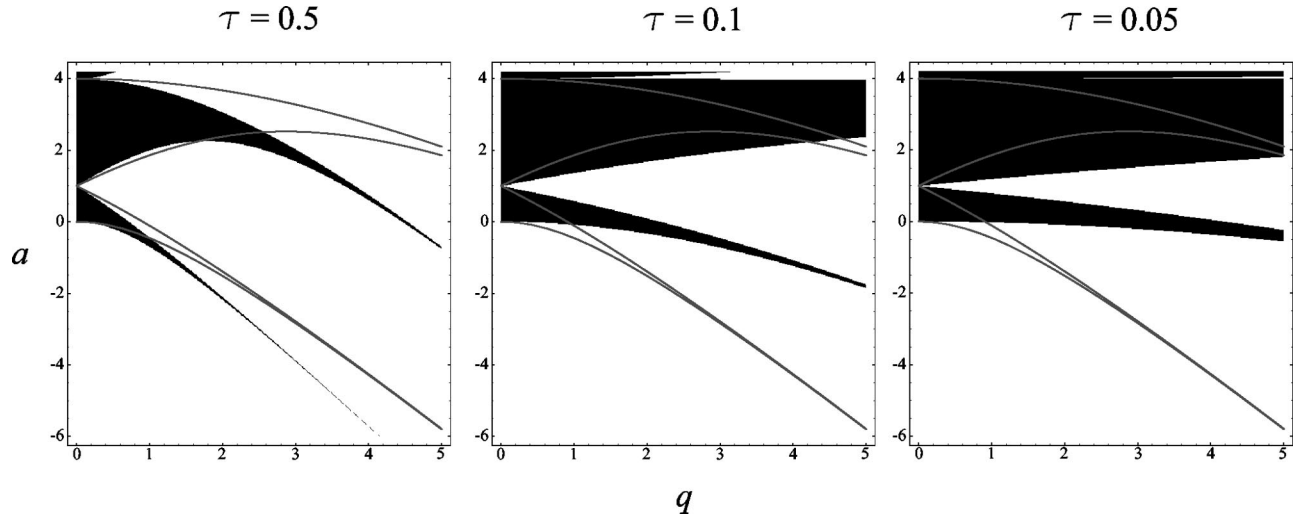

FIG. 4. The first two stability regions (shaded areas) with $\tau = 0.5, 0.1, 0.05$ for a pulsed voltage excitation. The stability regions for sinusoidal voltage excitation are shown for comparison (lines).

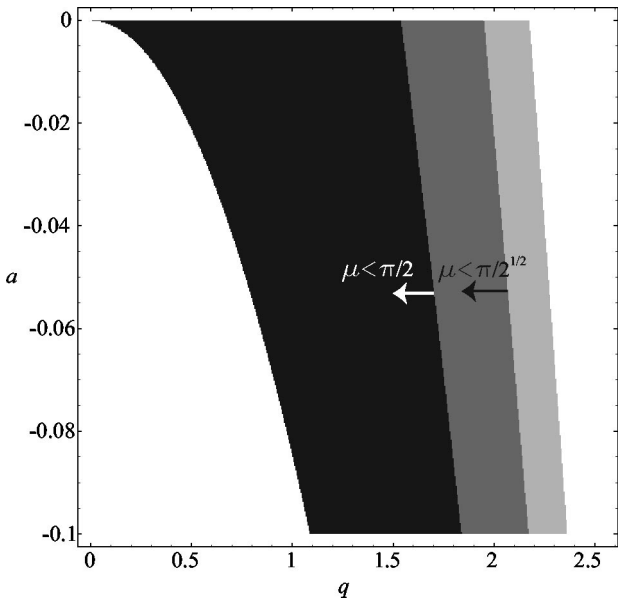

FIG. 5. Part of the first stabilty region for a pulsed voltage excitation of duration $\tau=0.1$ when $U=0$. The subregions subject to the conditions $\mu<\pi/\sqrt{2}$ and $\mu<\pi/2$ on the phase advance per wave form cycle are shown.

been done routinely in both storage rings[6] and traps,[10] one can increase the plasma density and it can be studied experimentally whether or not Coulomb crystals will form in the strong focusing environment of a pulse-excited trap. As a Coulomb crystal represents the ultimate in density with an interparticle interaction energy which is much larger than the ion kinetic energy, the regions of stability will be modified for this strongly coupled one component non-neutral plasma. A well-established result from the theory of crystalline beams states that the ring lattice periodicity should be at least $2\sqrt{2}$ as high as the maximum betatron (bare) tune. Stated in another way, we have the following criterion for the phase advance μ_u per unit cell (or rather per wave form cycle in the trap case)

$$\mu_u=\arccos\left[\frac{1}{2}\mathrm{Tr}\,M_u(\xi|\xi+\pi)\right]<\frac{\pi}{\sqrt{2}}. \tag{14}$$

The criterion Eq. (14) has been described as condition to avoid a parametric resonance between the phonon modes of the crystalline structure and the machine lattice periodicity[11,20] It should be stressed, however, that the maintenance condition is *not* limited to crystalline beams but is part of a far more general result[21,22] on the envelope stability of charged particle beams ranging from the emittance dominated (hot) beam, where the condition is $\mu_u<\pi/2$, to the space-charge dominated regime described by Eq. (14). In Fig. 5, the part of the first $\tau=0.1$ stability region for which Eq. (14) is fulfilled is shown along with the limit $\mu_u<\pi/2$. The latter criterion is a widely adopted condition for a smooth approximation treatment of strong focusing to be valid.[1] For a linear Paul trap operated at (q,a) parameters belonging to the stability region bounded by Eq. (14) it is expected that Coulomb crystals can be formed when trapped ions are subjected to laser cooling. We will test this experimentally in the near future in the case of $^{24}Mg^{+}$ and hopefully provide experimental support to the theory of crystalline ion beams.

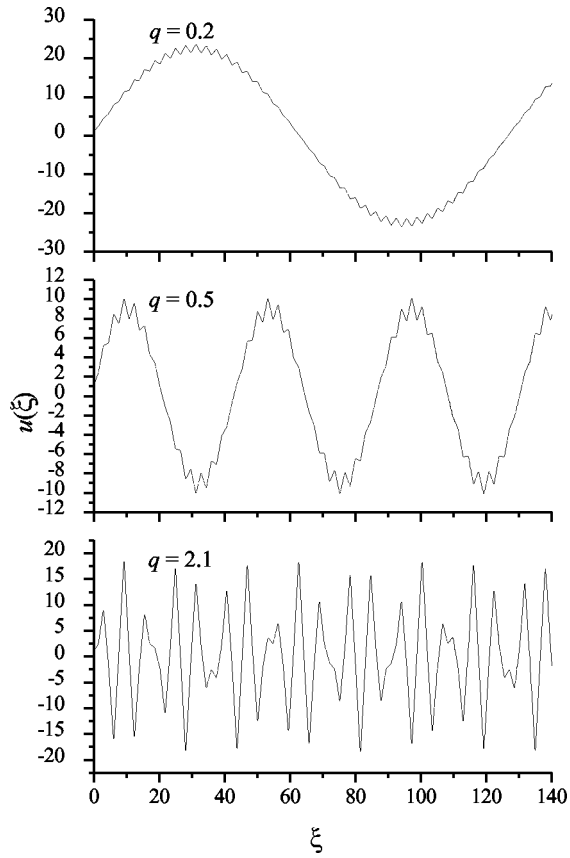

FIG. 6. Examples of particle trajectories for $q=0.2,0.5,2.1$, when $a=-10^{-3}$ and $\tau=0.1$.

IV. CONCLUSION

In conclusion, it is possible to build a bridge between the formalism of storage ring dynamics and that of an ion trap. The close analogy between the transverse dynamics of storage rings and a pulse-excited linear Paul trap can be exploited to test theoretical results on the stability of crystalline ion beams with respect to the maintenance condition concerning the maximum phase advance per unit cell. This can be done in table top size experiments and should provide valuable information before commissioning a large scale specialized storage ring for crystalline ion beams. We have outlined how the stability criterions can be stated in the (q,a)-parameter formalism usually applied by the ion trap community. This will serve as a reference frame for experimental investigations on crystalline beam stability using a linear Paul trap.

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APPENDIX: TRANSFER MATRIX

Given the initial conditions $[u(\xi_0),\dot{u}(\xi_0)]$ at time ξ_0 for Eq. (9), propagation to a later time ξ can be obtained using matrix multiplication via Eq. (11). Considering, for example, $\xi_0=\frac{1}{2}\tau\pi$ and using the transformation $\tilde{\xi}=\xi-\xi_0$ we have (leaving out subscript u)

$$M(\tilde{\xi}|0)=S(\tilde{\xi}-\lfloor\tilde{\xi}/\pi\rfloor)S^{\lfloor\tilde{\xi}/\pi\rfloor}(\pi), \tag{A1}$$

where

$$S(\Xi)=\begin{cases} T(a,\Xi) & \text{if} \quad 0<\Xi\leqslant\left(\frac{1}{2}-\tau\right)\pi \\ T\left(a+2q,\Xi-\left[\frac{1}{2}-\tau\right]\pi\right)S\left(\left[\frac{1}{2}-\tau\right]\pi\right) & \text{if} \quad \left(\frac{1}{2}-\tau\right)\pi<\Xi\leqslant\frac{1}{2}\pi \\ T\left(a,\Xi-\frac{1}{2}\pi\right)S\left(\frac{1}{2}\pi\right) & \text{if} \quad \frac{1}{2}\pi<\Xi\leqslant(1-\tau)\pi \\ T(a-2q,\Xi-[1-\tau]\pi)S([1-\tau]\pi) & \text{if} \quad (1-\tau)\pi<\Xi\leqslant\pi, \end{cases} \tag{A2}$$

and

$$T(K,l)=\tfrac{1}{2}\begin{bmatrix} e^{\sqrt{-K}l}+e^{-\sqrt{-K}l} & \frac{1}{\sqrt{-K}}(e^{\sqrt{-K}l}-e^{-\sqrt{-K}l}) \\ \sqrt{-K}(e^{\sqrt{-K}l}-e^{-\sqrt{-K}l}) & e^{\sqrt{-K}l}+e^{-\sqrt{-K}l} \end{bmatrix} \quad \text{if} \quad K\neq 0, \tag{A3}$$

$$T(K,l)=\begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} \quad \text{if} \quad K=0.$$

In Fig. 6, examples of single particle trajectories obtained by using the propagation matrix Eq. (A1) are shown.

[1] M. Reiser, *Theory and Design of Charged Particle Beams* (Wiley, New York, 1994).
[2] R. C. Davidson, *Physics of Nonneutral Plasmas* (Addison–Wesley, Reading, MA, 1990).
[3] H. Okamoto and H. Tanaka, Nucl. Instrum. Methods Phys. Res. A **437**, 178 (1999).
[4] R. C. Davidson, H. Qin, and G. Shvets, Phys. Plasmas **7**, 1020 (2000).
[5] J. P. Schiffer and P. Kienle, Z. Phys. A **321**, 181 (1985).
[6] N. Madsen, P. Bowe, M. Drewsen, L. H. Hornekær, N. Kjærgaard, A. Labrador, J. S. Nielsen, J. P. Schiffer, P. Shi, and J. S. Hangst, Phys. Rev. Lett. **83**, 4301 (1999).
[7] R. W. Hasse, Phys. Rev. Lett. **83**, 3430 (1999).
[8] G. Birkl, S. Kassner, and H. Walther, Nature (London) **357**, 310 (1992).
[9] J. N. Tan, J. J. Bollinger, B. Jelenkovic, and D. J. Wineland, Phys. Rev. Lett. **75**, 4198 (1995).
[10] M. Drewsen, C. Brodersen, L. Hornekær, J. S. Hangst, and J. P. Schiffer, Phys. Rev. Lett. **81**, 2878 (1998).
[11] J. Wei, H. Okamoto, and A. M. Sessler, Phys. Rev. Lett. **80**, 2606 (1998).
[12] T. Schätz, D. Habs, C. Podlech, J. Wei, and U. Schramm, AIP Conf. Proc. **457**, 269 (1999).
[13] N. Kjærgaard and M. Drewsen, Phys. Lett. A **260**, 507 (1999).
[14] E. D. Courant, M. S. Livingston, and H. S. Snyder, Phys. Rev. **88**, 1190 (1952).
[15] W. Paul and H. Steinwedel, Z. Naturforsch. A **8**, 448 (1953).
[16] M. Abramowitz and I. A. Stegun, *Handbook of Mathemathical Functions* (Dover, New York, 1970).
[17] J. A. Richards, R. M. Huey, and J. Hiller, Int. J. Mass Spectrom. Ion Phys. **12**, 317 (1973).
[18] E. D. Courant and H. S. Snyder, Ann. Phys. (N.Y.) **3**, 1 (1958).
[19] M. G. Raizen, J. M. Gilligan, J. C. Bergquist, W. M. Itano, and D. J. Wineland, Phys. Rev. A **45**, 6493 (1992).
[20] X.-P. Li, A. M. Sessler, and J. Wei, in *Proceedings of the Fourth European Particle Accelerator Conference*, edited by D. Dubin and D. Schneider (World Scientific, Singapore, 1994), Vol. 2, p. 1379.
[21] S. Y. Lee and H. Okamoto, Phys. Rev. Lett. **80**, 5133 (1998).
[22] A. Riabko, M. Ellison, X. Kang, S. Y. Lee, D. Li, J. Y. Liu, X. Pei, and L. Wang, Phys. Rev. E **51**, 3529 (1995).