Two fermions in a 1D harmonic oscillator with strong interparticle interactions

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I. EXERCISES

Consider the following Hamiltonian in one dimension for two particles in a harmonic oscillator potential

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2 x_1^2 + \frac{1}{2}m\omega^2 x_2^2 + g\delta(x_1 - x_2) = H_1 + H_2 + g\delta(x_1 - x_2), \tag{1}$$

where x_1 and x_2 are the coordinates of the particles, p_1 and p_2 the momenta, m is the mass, and ω is the angular frequency of the oscillator trap. The interaction between the particles is given by a zero-range Dirac delta-function with strength g.

1) First consider the non-interacting case, $g = 0$. Sketch the potential and the eigenenergies for a single particle on the same plot. What are the wave functions for a single particle in the ground state and in the first excited state?

2) What is the ground-state energy if you have two identical bosons? What is the ground state energy for two identical fermions? What is the ground state for fermions with two opposite spin states (spin up and spin down)?

3) What does the $g = 0$ wave function look like for two identical fermions? Make a sketch of it using relative coodinates $x = x_1 - x_2$. What changes when we have $g \neq 0$?

4) Consider instead two fermions with different spin states, i.e. an up and a down spin pair. Write down the wave function for $g = 0$. Change to relative coordinates, $x = x_1 - x_2$ and $X = (x_1 + x_2)/2$, throw away the center-of-mass X coordinate (we have to worry about that for interactions), and sketch the wave function as a function of relative coordinate, x.

5) Now consider repulsive interactions, $g > 0$, and the case of two fermions with opposite spins. What happens to the energy of the state consider in **4)**? Make a sketch of the energy.

6) Argue that for a Dirac delta function interaction $g\delta(x_1 - x_2)$, the wave function is continuous but the derivative is not.

7) What is the energy of a state with one spin up in the ground state and one spin down in the first excited state? Compare this energy to the energy of two identical fermions from **2)**.

8) Combining your knowledge from **3)**, **5)** and **6)**, make a sketch of the energy of two identical fermions and of two fermions with opposite spins as the strength of the interaction goes from $g = 0$ to $g = +\infty$.

You now know what people mean when they say fermionization of two fermions with opposite spin states. This has been explored experimentally in G. Zürn *et al.*, Phys. Rev. Lett. **108**, 075303 (2012). What remains is to figure out what the wave function looks like at $g = +\infty$ for the two fermions with different spins.

9) The Hamiltonian conserves parity and we can thus classify state as even or odd under parity. What is the parity of the wave function in **3)**? What is the parity of the wave function in **4)**?

10) The wave function must be a solution of the non-interacting ($g = 0$) Hamiltonian expect when $x = x_1 - x_2 = 0$ where the delta function contributes. Argue that for $g = +\infty$, the wave function at $x = 0$ has to vanish.

11) Use conservation of parity to argue that the ground state wave function for two fermions with opposite spins at $g = \infty$ must have even parity.

12) Combine non-interacting wave functions for $x < 0$ and $x > 0$ for two fermions with different spins into an even parity solution that vanishes at $x = 0$.